Unitary representations of $\mathcal{SW}(3/2,2)$ superconformal algebra

Doron Gepner, Boris Noyvert

Department of Particle Physics,

Weizmann Institute of Science,

76100, Rehovot, Israel.

Abstract

$\mathcal{SW}(3/2,2)$ superconformal algebra is $\mathcal{W}$ algebra with two Virasoro operators. The Kac determinant is calculated and the complete list of unitary representations is determined. Two types of extensions of $\mathcal{SW}(3/2,2)$ algebra are discussed. A new approach to construction of $\mathcal{W}$ algebras from rational conformal field theories is proposed.

1 Introduction

After the fundamental work of Zamolodchikov [1] conformal $\mathcal{W}$ algebras become the subject of great interest in many branches of physics and mathematics. Many $\mathcal{W}$ algebras are known (for review see [2]). However there is no complete classification of $\mathcal{W}$ algebras, and representation theory only of some of them is developed [3, 4, 5]. The study of minimal models [6] of $\mathcal{W}$ algebras is relevant for the classification problem of rational conformal field theories.

In this paper we study $\mathcal{SW}(3/2,2)$ extended superconformal algebra, generated by two supercurrents of dimensions $\frac{3}{2}$ and 2. The algebra was first constructed in Ref. [7].

At central charge $c = 12$ the algebra has geometrical meaning, it is associated to 8-dimensional manifolds of $Spin(7)$ holonomy [8, 9]. These manifolds are relevant for string theory. Sigma models based on $Spin(7)$ holonomy manifolds can provide a superstring compactification scheme down to 2 dimensions, leaving minimal number of space–time supersymmetries. In such a compactification $\mathcal{SW}(3/2,2)$ algebra plays the same role, as $N = 2$ superconformal algebra in compactifications on Calabi–Yau manifolds.

Our main subject is the representation theory of $\mathcal{SW}(3/2, 2)$ algebra. We define highest weight representations of the algebra, calculate their Kac determinant, study conditions of unitarity and list all the unitary representations (Appendix B).

The two supercurrents contain four conformal fields of scaling dimensions $\frac{3}{2}$, 2, 2, $\frac{5}{2}$. Two dimension 2 generators can be represented as two commuting Virasoro operators. One of these Virasoro subalgebras is in the rational regime, since its central charge is less than 1. This is a crucial point in the study of unitary representation theory of $\mathcal{SW}(3/2, 2)$ algebra. Unitarity is constrained to two series of models at central charges

$$
\begin{aligned}
c_p^{(1)} &= 6 - \frac{18}{p+1} \\
c_p^{(2)} &= 6 + \frac{18}{p}
\end{aligned}
\qquad p = 3, 4, 5 \ldots
\tag{1.1}
$$

together with their limiting point $c = 6$. There is no continuum of allowed by unitarity central charges. (In some sense, the continuum shrinks to the point $c = 6$.)

At fixed central charge from (1.1) the unitary spectrum consists of discrete points and continuous lines in the two dimensional space of weights. In particular, any representation with nonnegative scaling dimension ($h \geq 0$) is allowed. In Section 8 we decompose unitary highest weight representations of $\mathcal{SW}(3/2, 2)$ algebra to representations of the rational Virasoro subalgebra. The decomposition involves all the representations from one row (at $c = c_p^{(1)}$) or from one column (at $c = c_p^{(2)}$) of the Kac table of the Virasoro algebra minimal model. A subset of the discrete spectrum representations forms minimal models of $\mathcal{SW}(3/2, 2)$ algebra (Section 9).

In order to eliminate the continuous spectrum one should consider extensions of $\mathcal{SW}(3/2, 2)$ algebra. Two types of extensions of $\mathcal{SW}(3/2, 2)$ algebra are constructed in Section 10, relating $\mathcal{SW}(3/2, 2)$ to two other extended superconformal algebras $\mathcal{SW}(3/2, 3/2, 2)$ [10] and $\mathcal{SW}^2(1, 2)$ [11]. (See [2] for notations.) The second extension leads to interesting class of models possessing $N = 2$ superconformal symmetry.

The paper is organized as follows. Section 2 introduces the operator product expansions (OPEs) of $\mathcal{SW}(3/2, 2)$ algebra. The bosonic subalgebra is decomposed to two commuting Virasoro algebras in Section 3.

In Section 4 we propose a new way of construction of $\mathcal{SW}(3/2, 2)$ algebra from minimal models of the Virasoro algebra. The similar approach can be applied to construction of new $\mathcal{W}$ algebras.

The highest weight representations of $\mathcal{SW}(3/2, 2)$ are defined in Section 5. There are two sectors: Neveu–Schwarz (NS) and Ramond, each is labeled by two weights. Coulomb gas representation is redeveloped in Section 6, and the Kac determinant is calculated in Section 7.

2

The restrictions of unitarity on highest weight representations are analyzed in Section 8. We illustrate the calculations by two examples: $c = 12/5$ and $c = 12$ models. Fusion rules are discussed in Section 9 and extensions of $\mathcal{SW}(3/2, 2)$ algebra – in Section 10.

Most of the work is done with a help of *Mathematica* package [12] for symbolic computation of operator product expansions.

2 $\mathcal{SW}(3/2, 2)$ algebra

Motivated by interplay between string theory and geometry Shatashvili and Vafa [8, 9] studied extended symmetry algebras, which underlie sigma models with $N = 1$ superconformal symmetry on manifolds of exceptional holonomy, namely 7–dimensional manifolds of G_2 holonomy and 8–dimensional manifolds of $Spin(7)$ holonomy.

The symmetry algebra contains Virasoro generator T and supersymmetry generator G. In the $Spin(7)$ holonomy case the existence of $Spin(7)$ invariant 4–form leads to dimension 2 operator $\tilde{X}$ [8]. It has a superpartner $\tilde{M}$ of dimension $\frac{5}{2}$. These four operators constitute a closed conformal algebra with central charge $c = 12$. The subalgebra generated by $\tilde{X}$ coincides with Ising model, i.e. Virasoro algebra with central charge $\frac{1}{2}$.

As pointed out in [13] the algebra is a special ($c = 12$) case of the generic algebra, which was obtained in Ref. [7] using a conformal bootstrap method and was called by authors super W_2 algebra. We adopt the conventions of [2] and call it $\mathcal{SW}(3/2, 2)$ algebra.

The algebra is an extension of $N = 1$ superconformal algebra by dimension 2 superconformal multiplet, which consists of Virasoro primary field W of scaling dimension $\Delta_W = 2$ and its superpartner U of dimension $\Delta_U = \frac{5}{2}$:

$$G(z) \, W(w) = \frac{U(w)}{z - w} \,, \tag{2.1}$$

$$G(z) \, U(w) = \frac{2 \, \Delta_W \, W(w)}{(z - w)^2} + \frac{\partial W(w)}{z - w} \,. \tag{2.2}$$

Our normalization of U differs from one in Ref. [7] by factor $-i\sqrt{5}$. The OPE of W with itself is

$$W(z) \, W(w) = \frac{c/2}{(z - w)^4} + \frac{2 \, T(w) + \frac{2 \, (6 + 5 \, c)}{\sqrt{15 - c} \, \sqrt{21 + 4 \, c}} \, W(w)}{(z - w)^2} +$$
$$+ \, \frac{\partial T(w) + \frac{6 + 5 \, c}{\sqrt{15 - c} \, \sqrt{21 + 4 \, c}} \, \partial W(w)}{z - w} \,. \tag{2.3}$$

The important property of the algebra that it closes nonlinearly: the singular terms of OPEs $W(z)\,U(w)$ and $U(z)\,U(w)$ contain normal ordered products of basic fields. The OPEs are given in Appendix A.

The mode expansions of generators are defined in the usual way:

$$\mathcal{O}(z) = \sum_n \mathcal{O}_n\, z^{-n-\Delta_{\mathcal{O}}}\,, \tag{2.4}$$

where $\mathcal{O}$ is any generator of the algebra, and $\Delta_{\mathcal{O}}$ is its scaling dimension. (The modes of $T(z)$ are denoted by L_n.) Unitarity is introduced by standard conjugation relation $\mathcal{O}_n^\dagger = \mathcal{O}_{-n}$ for any generator except U. $\mathcal{SW}(3/2,2)$ commutation relations are consistent with nonstandard conjugation property $U_n^\dagger = -U_{-n}$.

3 Virasoro algebra embeddings

The algebra contains a few different subalgebras. These are Virasoro subalgebra generated by $T(z)$, superconformal subalgebra generated by $T(z)$ and $G(z)$, and bosonic subalgebra generated by $T(z)$ and $W(z)$. Let's discuss the last one.

One can introduce instead of T and W two new dimension 2 operators:

$$\begin{aligned} T &= A + B\,, \\ W &= \sqrt{\tfrac{21+4\,c}{15-c}}\,A - \sqrt{\tfrac{15-c}{21+4\,c}}\,B\,. \end{aligned} \tag{3.1}$$

As a result the bosonic part of $\mathcal{SW}(3/2,2)$ is decomposed to two commuting Virasoro algebras A and B with central charges

$$c_A = \frac{c\,(15-c)}{3\,(12+c)}\,, \qquad c_B = \frac{c\,(21+4\,c)}{3\,(12+c)}\,. \tag{3.2}$$

We will call the A and B Virasoro algebras the internal and external algebras.

Unitary representations of an algebra are necessarily unitary representations of all its subalgebras. In the region $c \geq 0$ we have $0 \leq c_A \leq 1$. Therefore by applying the nonunitarity theorem [14, 15] to A Virasoro subalgebra we deduce that there are no unitary representations of $\mathcal{SW}(3/2,2)$ out of special values of central charge $c_A = 1$ and

$$c_A(c) = 1 - \frac{6}{p(p+1)}\,, \qquad p = 3,4,5\ldots \tag{3.3}$$

In the latter case (3.3) there is a finite number of unitary highest weight representations of internal Virasoro algebra: $\Phi_{m,n}$ [6]. Their weights form Kac table:

$$a_{m,n} = \frac{(m\,p - n\,(1+p))^2 - 1}{4\,p\,(1+p)}\,, \qquad m = 1,\ldots,p\,, \quad n = 1,\ldots,p-1\,. \tag{3.4}$$

Resolving equation (3.3) for c we get two series of the unitary models (1.1). The series $c_p^{(1)}$ and $c_p^{(2)}$ start from $3/2$ and 12 and converge to $c = 6$ from below and from up respectively. The limiting point $c = 6$ corresponds to the value $c_A = 1$ and also is not forbidden by the nonunitarity theorem.

Other subalgebras give no new restrictions on unitarity. Indeed in the range $3/2 \le c \le 12$ the representations of superconformal algebra are unitary. And since $c_B \ge 1$ in this range the B subalgebra is also unitary.

As one could expect the $c = 3/2$ model can be written in terms of one free boson and one free fermion:

$$
\begin{aligned}
&A = \tfrac{1}{2}{:}\psi\partial\psi{:} && B = -\tfrac{1}{2}{:}\partial\phi\partial\phi{:} \\
&G = {:}\psi\partial\phi{:} && U = -\sqrt{2}{:}\partial\psi\partial\phi{:} + \tfrac{1}{\sqrt{2}}{:}\psi\partial^2\phi{:}
\end{aligned}
\tag{3.5}
$$

where the OPEs of the free fields are given by

$$
\phi(z)\,\phi(w) = -\log(z - w)\,,
\tag{3.6}
$$

$$
\psi(z)\,\psi(w) = -\frac{1}{z - w}\,.
\tag{3.7}
$$

In this simple example the fermionic part of the stress-energy tensor is the internal Virasoro operator and its bosonic part is the external Virasoro operator. The internal Virasoro part is identical to the Ising model.

4 Different construction of $\mathcal{SW}(3/2, 2)$ algebra

$\mathcal{SW}(3/2, 2)$ algebra was first constructed in Ref. [7] by conformal bootstrap method. In this section we present another (probably, less formal) way of constructing the algebra.

As we have seen in the previous section, the bosonic part of $\mathcal{SW}(3/2, 2)$ splits to two commuting Virasoro algebras, A and B. Other generators of the algebra should fall in some representation of $A \oplus B$ algebra. Indeed $G(z)$ is a primary field of $A \oplus B$ with dimensions $(a_{1,2}, \frac{3}{2} - a_{1,2})$ at $c = c_p^{(1)}$ and $(a_{2,1}, \frac{3}{2} - a_{2,1})$ at $c = c_p^{(2)}$.

The starting point of the construction is Virasoro algebra A in the minimal model regime, i.e. the value of central charge c_A is given by (3.3). We want to extend the algebra to include the minimal model primary field $\Phi_{1,2}$. From other point of view we want this field to be superconformal generator of dimension $\frac{3}{2}$. This is possible, if A is not the full Virasoro operator, but there is another Virasoro algebra B (commuting with A) which completes the full Virasoro operator $T = A + B$. Now we can introduce the field $G(z)$, which is primary with respect to A with weight $a_{1,2}$ and primary with respect to T with weight $\frac{3}{2}$. The OPEs are:

$$
T(z)\,G(w) = \frac{\frac{3}{2}\,G(w)}{(z - w)^2} + \frac{\partial G(w)}{z - w}\,,
\tag{4.1}
$$

$$A(z)\,G(w) = \frac{a_{1,2}\,G(w)}{(z-w)^2} + \frac{(A_{-1}G)(w)}{z-w}\,. \tag{4.2}$$

$(A_{-1}G)$ is a new generator of the algebra of the full dimension $\frac{5}{2}$. The OPE of $A(z)$ with $(A_{-1}G)(w)$ includes the field $(A^2_{-1}G)(w)$, but this is not a new generator due to existence of null vector on level 2 of $\Phi_{1,2}$ representation of A Virasoro algebra:

$$\left(A^2_{-1} - \frac{2+4\,a_{1,2}}{3}\,A_{-2}\right)\Phi_{1,2} = 0\,. \tag{4.3}$$

So $(A^2_{-1}G)$ is proportional to $(A_{-2}G)$, which can be written as $:AG:$. We need to introduce only one new field U:

$$U = \frac{1}{N_U}\left((A_{-1}G) - \frac{2}{3}\,a_{1,2}\,\partial G\right). \tag{4.4}$$

We add ∂G term in order to make U field primary with respect to T, and N_U is just a normalization factor. With this definition we write the OPEs:

$$A(z)\,G(w) = \frac{a_{1,2}\,G(w)}{(z-w)^2} + \frac{N_U\,U(w) + \frac{2}{3}\,a_{1,2}\,\partial G(w)}{z-w}\,, \tag{4.5}$$

$$T(z)\,U(w) = \frac{\frac{5}{2}\,U(w)}{(z-w)^2} + \frac{\partial U(w)}{z-w}\,, \tag{4.6}$$

$$A(z)\,U(w) = \frac{\frac{2}{N_U}\,a_{1,2}\left(1-\frac{2}{3}\,a_{1,2}\right)G(w)}{(z-w)^3} +$$
$$+ \frac{\left(1+\frac{a_{1,2}}{3}\right)U(w) + \frac{2}{3\,N_U}\,a_{1,2}\left(1-\frac{2}{3}\,a_{1,2}\right)\partial G(w)}{(z-w)^2} +$$
$$+ \frac{\frac{2}{3\,N_U}\left(1+2\,a_{1,2}\right):AG:(w) - \frac{2}{3}\,a_{1,2}\,\partial U(w) - \frac{4}{9\,N_U}\,a^2_{1,2}\,\partial^2 G(w)}{z-w}\,. \tag{4.7}$$

This set of OPEs guarantees that Jacobi identities of type $[X,X,Y]$ are satisfied, where X stands for T or A, Y – for G or U.

Now we need to close the algebra by specifying the OPEs of type $Y(z)Y(w)$. From the point of view of A Virasoro algebra G and U are fields in $\Phi_{1,2}$ representation, A and T are in $\Phi_{1,1}$ (vacuum) representation. The fusion [6]

$$\Phi_{1,2} \times \Phi_{1,2} = \Phi_{1,1} + \Phi_{1,3} \tag{4.8}$$

includes the new representation $\Phi_{1,3}$. It should enter to the OPEs of type $Y(z)Y(w)$. It is easy to check that fields $:G\partial G:$ and $:GU:$ are in the $\Phi_{1,3}$ representation. We construct $G(z)G(w)$, $G(z)U(w)$, $U(z)U(w)$ as most general OPEs including only A,

T operators with their derivatives and composites and $:G\partial G:$, $:GU:$. After that the arbitrary coefficients are fixed as functions of c and $a_{1,2}$ by Jacobi identities of type $[X, Y, Y]$. Jacobi identities of type $[Y, Y, Y]$ also fix the connection between the full central charge and $a_{1,2}$:

$$a_{1,2} = \frac{15 - c}{2\,(12 + c)}\,.\tag{4.9}$$

Finally we obtain $\mathcal{SW}(3/2, 2)$ algebra in the form presented in Appendix A.

We started from discrete unitary values of c_A, but the algebra can be continued smoothly to any $c_A \leq 1$, i.e. to any c. Another remark: one could start from $\Phi_{2,1}$ representation instead of $\Phi_{1,2}$, but the same algebra is obtained.

Highest weight representations of the algebra are introduced in the next section, but already at this stage we can predict some representations. There is a magic relation between the dimensions of any Kac table (3.4):

$$a_{1,n} + a_{m,1} - a_{m,n} = \frac{(m - 1)(n - 1)}{2} \qquad \forall p\,.\tag{4.10}$$

The right hand side of this equation is integer or half-integer. Taking into account that the relation (4.10) corresponds to the fusion rule [6]

$$\Phi_{1,n} \times \Phi_{m,1} = \Phi_{m,n}\,,\tag{4.11}$$

the fields $\Phi_{1,n}$ and $\Phi_{m,1}$ are found to be local or semilocal with respect to each other.

The $\mathcal{SW}(3/2, 2)$ generator G behaves as $\Phi_{1,2}$ under internal Virasoro symmetry. The purely internal representations of $A \oplus B$ algebra $(\Phi_{m,1}, 0)$ are local (odd m) or semilocal (even m) with respect to supercurrent G. They are first candidates to Neveu–Schwarz (NS) and Ramond representations respectively. Indeed we find them in the discrete unitary spectrum (see end of Section 8). Moreover this set of fields forms minimal models of $\mathcal{SW}(3/2, 2)$ algebra (Section 9).

The type of construction, discussed in this section, can be used for building new conformal algebras. For example, one can take $N = 1$ superconformal minimal model as internal algebra, and to extend it by simplest nontrivial NS field $\Phi_{1,3}$. We expect that $\mathcal{SW}(3/2, 3/2, 2)$ conformal algebra ([10] and Section 10 of this paper) can be constructed this way. But this is a subject to another paper.

The superconformal algebras of Refs. [16, 17] can also be treated as extensions of rational conformal models. The role of internal algebra is played by affine Lie algebra $\widehat{g}$. It is extended by supercurrents G_i in some (usually fundamental, see [16, 17]) representation of g. The dimension of supercurrents is $\frac{3}{2}$ with respect to the full Virasoro generator T, which can be seen as a sum of two parts: $T = A_g + B$, where A_g is Sugawara energy-momentum tensor of $\widehat{g}$ and B is external Virasoro operator, commuting with A_g and with all the currents of $\widehat{g}$. The algebra closes nonlinearly: the OPEs $G_i(z)\,G_j(w)$ involve normal orderings of the $\widehat{g}$ currents.

5 Highest weight representations

One can define consistently two modings of $\mathcal{SW}(3/2, 2)$ algebra. In both cases modes of the bosonic generators are integer. Modes of the fermionic generators can be chosen half integer (NS sector) or integer (Ramond sector). There are no other possible modings of the algebra.

5.1 NS sector

There are two zero modes: L_0 and A_0. Since their commutator is zero, they generate the Cartan subalgebra. Following notations of [8] we label the highest weight representations by eigenvalues of A_0 and B_0:

$$
\begin{aligned}
A_0 \big| a, b \big\rangle &= a \big| a, b \big\rangle, \\
B_0 \big| a, b \big\rangle &= b \big| a, b \big\rangle, \\
\mathcal{O}_n \big| a, b \big\rangle &= 0 \qquad \forall n > 0,
\end{aligned}
\tag{5.1}
$$

and $\mathcal{O}$ being any operator of the algebra. a is called internal dimension and b is called external dimension of the state. Also we will often use the full scaling dimension $h = a + b$, the eigenvalue of L_0.

The primary field $\Psi(z)$ corresponding to the highest weight state $\big| a, b \big\rangle$ has the following OPEs with generators of the algebra:

$$
\begin{aligned}
T(z)\,\Psi(w) &= \frac{h\,\Psi(w)}{(z-w)^2} + \frac{\partial \Psi(w)}{z-w} + \ldots \\
A(z)\,\Psi(w) &= \frac{a\,\Psi(w)}{(z-w)^2} + \frac{(A_{-1}\Psi)(w)}{z-w} + \ldots \\
G(z)\,\Psi(w) &= \frac{(G_{-1/2}\Psi)(w)}{z-w} + \ldots \\
U(z)\,\Psi(w) &= \frac{(U_{-1/2}\Psi)(w)}{(z-w)^2} + \frac{(U_{-3/2}\Psi)(w)}{z-w} + \ldots
\end{aligned}
\tag{5.2}
$$

At this point we are ready to write down the NS representations of the $c = 3/2$ model (3.5):

$$
\begin{array}{ll}
\big| 0, 0 \big\rangle & \text{vacuum} \\
\big| \frac{1}{2}, 0 \big\rangle & \psi(z) \\
\big| \frac{1}{16}, \frac{1}{16} \big\rangle & \sigma(z)\, s(z) \\
\big| 0, \frac{1}{2}\beta^2 \big\rangle & e^{i\beta\phi(z)}
\end{array}
\tag{5.3}
$$

where $\sigma(z)$ and $s(z)$ are twist fields of free fermion and free boson theory respectively. They are defined by OPEs (see e.g., [18]):

$$\psi(z)\,\sigma(w) \sim \frac{\mu(w)}{(z-w)^{1/2}} + \dots \tag{5.4}$$

$$\partial\phi(z)\,s(w) \sim \frac{\tau(w)}{(z-w)^{1/2}} + \dots \tag{5.5}$$

The dimensions of σ, μ, s, τ are $\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}$ respectively.

The representations (5.3) are obviously unitary by construction. The unitary spectrum contains continuous line: $\left|\,0\,,b\geq 0\,\right\rangle$ and two discrete points: $\left|\,\frac{1}{2}\,,0\,\right\rangle$ and $\left|\,\frac{1}{16}\,,\frac{1}{16}\,\right\rangle$. As we will see all other unitary models also consist of discrete states and continuum.

5.2   Ramond sector

The highest weight state is annihilated by all the operators with positive indices. The zero mode algebra here is more complicated than in the NS sector. There are four zero modes: L_0, A_0, G_0, U_0. L_0 is commutative with three other zero modes, so it can be represented by a number h. The commutation relations between A_0, G_0, U_0 are nonlinear, they contain normal ordered products $:\!AG\!:_0$, $:\!TA\!:_0$, $:\!GU\!:_0$, $:\!G\partial G\!:_0$, which are infinite sums in terms of modes of generators (see appendix C for exact definitions). However, we always keep in mind that we apply these operators to highest weight states. Any positive mode acting on highest weight state gives zero. So in the highest weight representation only a few terms from the infinite sums will survive and the (anti)commutators of the zero modes will be:

$$
\begin{aligned}
[A_0, G_0] &= -\frac{\sqrt{15-c}\,\sqrt{21+4\,c}}{3\,(12+c)}\,U_0\,, \\[2mm]
[A_0, U_0] &= \frac{3\,\sqrt{15-c}\,(3+c)\,G_0}{4\,(12+c)\,\sqrt{21+4\,c}} + \frac{9\,U_0}{12+c} - \frac{54\,G_0\,A_0}{\sqrt{15-c}\,\sqrt{21+4\,c}}\,, \\[2mm]
\{G_0, G_0\} &= -\frac{c}{12} + 2\,h\,, \\[2mm]
\{G_0, U_0\} &= 0\,, \\[2mm]
\{U_0, U_0\} &= -\frac{3\,(3+c)\,(c-24\,h)}{16\,(21+4\,c)} \\[2mm]
&\quad + \frac{54\,G_0\,U_0}{\sqrt{15-c}\,\sqrt{21+4\,c}} + \frac{27\,(12+c)\,(c-24\,h)\,A_0}{2\,(15-c)\,(21+4\,c)}\,.
\end{aligned}
\tag{5.6}
$$

They define finite dimensional quadratic $\mathcal{W}$ superalgebra.

9

The irreducible representations of this algebra are two dimensional. We choose A_0 to be diagonal:

$$A_0 = \begin{pmatrix} a & 0 \\ 0 & a + \Delta a \end{pmatrix}, \tag{5.7}$$

where Δa is some function of a, h, c. Then G_0 and U_0 are:

$$G_0 = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}. \tag{5.8}$$

$(U_0^\dagger = -U_0)$. The commutation relations (5.6) are satisfied by

$$g = \sqrt{h - c/24} \quad \text{and} \quad u = g \, \Delta a \, \frac{3 \, (12 + c)}{\sqrt{15 - c} \, \sqrt{21 + 4 \, c}} . \tag{5.9}$$

There are two solutions for Δa:

$$\Delta a_{1,2} = \frac{9 \pm \sqrt{(6 - c)^2 + 72 \, a \, (12 + c)}}{2 \, (12 + c)} . \tag{5.10}$$

Note that Δa doesn't depend on h, it depends only on a and c.

Ramond sector is well defined only for $h \geq c/24$. In the special case $h = c/24$ the representation of the zero mode algebra (5.6) can be chosen one dimensional with

$$G_0 = U_0 = 0, \qquad A_0 = a . \tag{5.11}$$

We will call such representation Ramond ground state by analogy to $N = 2$ superconformal algebra.

Acting by negative modes of generators on the representation of the zero mode algebra we get the full tower of states of Ramond representation. We will denote Ramond ground highest weight states by $\big| a \, , b \big\rangle$ $(a + b = c/24)$ and two dimensional Ramond highest weight states by $\left| \begin{matrix} a_\uparrow \, , b_\uparrow \\ a_\downarrow \, , b_\downarrow \end{matrix} \right\rangle$, where $a_\uparrow$ and $a_\downarrow$ are two eigenvalues of A_0, $b_\uparrow$ and $b_\downarrow$ are two eigenvalues of B_0 $(a_\uparrow + b_\uparrow = a_\downarrow + b_\downarrow = h)$. $a_\uparrow$ and $a_\downarrow$ are not independent, their difference is fixed by $a_\downarrow - a_\uparrow = \Delta a_{1,2}$ from (5.10).

The OPEs of generators of $\mathcal{SW}(3/2, 2)$ with primary field of Ramond type are:

$$
\begin{aligned}
T(z) \, \Psi(w) &= \frac{h \, \Psi(w)}{(z - w)^2} + \frac{\partial \Psi(w)}{z - w} + \dots \\
A(z) \, \Psi(w) &= \frac{(A_0 \Psi)(w)}{(z - w)^2} + \frac{(A_{-1} \Psi)(w)}{z - w} + \dots \\
G(z) \, \Psi(w) &= \frac{(G_0 \Psi)(w)}{(z - w)^{3/2}} + \frac{(G_{-1} \Psi)(w)}{(z - w)^{1/2}} + \dots \\
U(z) \, \Psi(w) &= \frac{(U_0 \Psi)(w)}{(z - w)^{5/2}} + \frac{(U_{-1} \Psi)(w)}{(z - w)^{3/2}} + \dots
\end{aligned}
\tag{5.12}
$$

10

Coming back to the $c = 3/2$ example (3.5) we write unitary Ramond representations of the model:

$$
\begin{array}{ll}
\left|\frac{1}{2},\frac{1}{16}\right\rangle & \left(\begin{array}{c}\psi(z)\,s(z)\\ \tau(z)\end{array}\right)\\[2pt]
\left|0\,,\frac{9}{16}\right\rangle & \\[2pt]
\left|0\,,\frac{1}{16}\right\rangle & s(z)\\[2pt]
\left|\frac{1}{16}\,,0\right\rangle & \sigma(z)\\[2pt]
\left|\frac{1}{16}\,,\frac{1}{2}\beta^2\right\rangle & \left(\begin{array}{c}\sigma(z)\\ \mu(z)\end{array}\right)\mathrm{e}^{\mathrm{i}\beta\phi(z)}\\[2pt]
\left|\frac{1}{16}\,,\frac{1}{2}\beta^2\right\rangle &
\end{array}
\tag{5.13}
$$

with twist fields defined in (5.4) and (5.5).

6 Free field representation

The free field representation of $\mathcal{SW}(3/2,2)$ algebra was derived by several authors [19, 20]. They get the covariant representation, i.e. in terms of two $N = 1$ free superfields: $\phi + \theta\,\psi$. We don't introduce here the superspace formalism but use explicit noncovariant formulation.

The construction of $N = 1$ superalgebra in terms of free bosons and fermions is well known:[1]

$$
\begin{aligned}
T =\ & \frac{1}{2}\,:\!\psi_1\partial\psi_1\!:\,+\,\frac{1}{2}\,:\!\psi_2\partial\psi_2\!:\\
& -\frac{1}{2}\,:\!\partial\phi_1\partial\phi_1\!:\,-\,\frac{1}{2}\,:\!\partial\phi_2\partial\phi_2\!:\,-\,\frac{\mathrm{i}\,(6-c)}{6\,\sqrt{3+c}}\,\partial^2\phi_1 - \frac{3-2\,c}{6\,\sqrt{3+c}}\,\partial^2\phi_2\,,\\
G =\ & :\!\psi_1\partial\phi_1\!:\,+\,:\!\psi_2\partial\phi_2\!:\,+\,\frac{\mathrm{i}\,(6-c)}{3\,\sqrt{3+c}}\,\partial\psi_1 + \frac{3-2\,c}{3\,\sqrt{3+c}}\,\partial\psi_2\,,
\end{aligned}
\tag{6.1}
$$

where the free fields are defined by (3.6) and (3.7).

We look for W being a linear combination (with c–dependent coefficients) of all possible dimension 2 composite fields constructed from the free fields. W must satisfy the OPEs with T, G and itself, from these conditions we calculate the coefficients. The result is:

$$
\begin{aligned}
W =\ & \frac{1}{\sqrt{15-c}\,\sqrt{21+4\,c}}\Bigg((3-2\,c)\,:\!\partial\phi_1\partial\phi_1\!:\,+\,\tfrac{1}{2}\,(15-c)\,:\!\partial\phi_2\partial\phi_2\!:\\
& -9\,\sqrt{3+c}\,:\!\psi_1\psi_2\partial\phi_1\!:\,-\,(3-2\,c)\,:\!\psi_1\partial\psi_1\!:\,+\,(15-c)\,:\!\psi_2\partial\psi_2\!:\\
& +3\,\mathrm{i}\,(6-c)\,:\!\psi_2\partial\psi_1\!:\,+\,\frac{\mathrm{i}\,(6-c)\,(3-2\,c)}{3\,\sqrt{3+c}}\,\partial^2\phi_1 + \frac{(15-c)\,(3-2\,c)}{6\,\sqrt{3+c}}\,\partial^2\phi_2 \Bigg)\,.
\end{aligned}
\tag{6.2}
$$

---

[1]Usually the basis in the space of bosons is chosen in such a way, that there is no $\partial^2\phi_2$ term in the expression for T. We choose rotated basis for later convenience.

The free field representation of U is obtained automatically from OPE (2.1).

Primary fields in NS sector are represented by free field exponentials:

$$V_{\vec{\beta}}(z) = \,: \exp\left(i\vec{\beta}\cdot\vec{\phi}(z)\right):\,, \qquad \vec{\beta}\cdot\vec{\phi} = \beta_1\,\phi_1 + \beta_2\,\phi_2\,. \tag{6.3}$$

Applying $T(z)$ and $A(z)$ to the exponentials we get the weights parameterized by

$$h(\vec{\beta}) = \frac{1}{2}\,\beta_1\left(\beta_1 + \frac{6-c}{3\sqrt{3+c}}\right) + \frac{1}{2}\,\beta_2\left(\beta_2 - i\,\frac{3-2c}{3\sqrt{3+c}}\right), \tag{6.4}$$

$$a(\vec{\beta}) = \frac{3+c}{2\,(12+c)}\,\beta_1\left(\beta_1 + \frac{6-c}{3\sqrt{3+c}}\right). \tag{6.5}$$

In Ramond sector there are also fermionic zero modes. We can choose two Pauli matrices σ_x and σ_y to represent ψ_1 and ψ_2 zero modes. A_0 and L_0 are diagonal in such a representation. So the Ramond states are constructed from two dimensional spin space and free field exponentials:

$$\begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} V_{\vec{\beta}}(z)\,. \tag{6.6}$$

Both states have the same full dimension:

$$h(\vec{\beta}) = \frac{1}{2}\,\beta_1\left(\beta_1 + \frac{6-c}{3\sqrt{3+c}}\right) + \frac{1}{2}\,\beta_2\left(\beta_2 - i\,\frac{3-2c}{3\sqrt{3+c}}\right) + \frac{1}{8}\,, \tag{6.7}$$

which differs from the expression in NS sector by $\frac{1}{8}$ ($\frac{1}{16}$ from each fermion). The internal dimensions of $|\uparrow\rangle$ and $|\downarrow\rangle$ states are

$$\begin{aligned} a(\vec{\beta},\uparrow) &= \frac{3+c}{2\,(12+c)}\left(\beta_1 + \frac{15-c}{6\sqrt{3+c}}\right)^2 - \frac{(6-c)^2}{72\,(12+c)}\,, \\ a(\vec{\beta},\downarrow) &= \frac{3+c}{2\,(12+c)}\left(\beta_1 - \frac{\sqrt{3+c}}{6}\right)^2 - \frac{(6-c)^2}{72\,(12+c)}\,. \end{aligned} \tag{6.8}$$

Note that in both sectors the expressions for dimensions have two independent $\mathbb{Z}_2$ symmetries:[2]

$$\beta_1 \to -\frac{6-c}{3\sqrt{3+c}} - \beta_1\,, \tag{6.9}$$

$$\beta_2 \to i\,\frac{3-2c}{3\sqrt{3+c}} - \beta_2\,. \tag{6.10}$$

One can identify 4 representations, related by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

---

[2]In Ramond sector the first symmetry interchanges $|\uparrow\rangle$ and $|\downarrow\rangle$ states.

7 Degenerate representations and Kac determinant

Degenerate representations of various conformal algebras were described in [21, 22, 23, 3, 4, 5, 24] using the technique of Coulomb gas formalism. We will follow the same methods.

Screening operators Q are operators which commute with all the algebra:

$$[Q, \mathcal{O}_n] = 0 \quad \forall n. \tag{7.1}$$

The form of Q

$$Q_{\vec{\alpha}} = \oint dz\, \Theta_{\vec{\alpha}}(z), \qquad \Theta_{\vec{\alpha}}(z) = \, :\vec{\alpha} \cdot \vec{\psi}(z)\, e^{i\,\vec{\alpha}\cdot\vec{\phi}(z)}:, \tag{7.2}$$

where $h(\vec{\alpha}) = \frac{1}{2}$, ensures the zero commutation relations with L_n and G_n. The condition $[Q, W_n] = 0$ fixes the vector $\vec{\alpha}$. There are 3 screening operators: $Q_{\vec{\alpha}_1}$, $Q_{\vec{\alpha}_2}$, $Q_{\vec{\alpha}_3}$, where

$$\begin{aligned}
\vec{\alpha}_1 &= \tfrac{3}{\sqrt{3+c}}\left(1,\, -i\right), \\
\vec{\alpha}_2 &= \tfrac{3}{\sqrt{3+c}}\left(-1,\, 0\right), \\
\vec{\alpha}_3 &= \tfrac{\sqrt{3+c}}{3}\left(1,\, 0\right).
\end{aligned} \tag{7.3}$$

The first two vectors (with omitted c-dependent coefficients) can be identified with simple roots of $osp(3|2) \approx B(1,1)$ Lie superalgebra, quantum Drinfeld–Sokolov reduction of which is exactly $\mathcal{SW}(3/2, 2)$ conformal algebra [19].

One can construct null states acting by screening operators on primary field:

$$\begin{aligned}
\eta_n = \left(Q_{\vec{\alpha}}\right)^n V_{\vec{\beta}-n\vec{\alpha}}(0) &= \\
= \oint_{C_1} dz_1\, \Theta_{\vec{\alpha}}(z_1) \cdots & \oint_{C_n} dz_n\, \Theta_{\vec{\alpha}}(z_n)\, V_{\vec{\beta}-n\vec{\alpha}}(0).
\end{aligned} \tag{7.4}$$

The state is a descendant of $V_{\vec{\beta}}$, but it behaves as $V_{\vec{\beta}-n\vec{\alpha}}$ under action of generators of the algebra:

$$\begin{aligned}
\mathcal{O}_k \left| \eta_n \right\rangle &= 0, \quad k > 0, \\
L_0 \left| \eta_n \right\rangle &= h(\vec{\beta} - n\,\vec{\alpha}) \left| \eta_n \right\rangle, \\
A_0 \left| \eta_n \right\rangle &= a(\vec{\beta} - n\,\vec{\alpha}) \left| \eta_n \right\rangle.
\end{aligned} \tag{7.5}$$

This means that the representation constructed from $V_{\vec{\beta}}$ is reducible and $\left| \eta_n \right\rangle$ must be a null vector on the level

$$N = h(\vec{\beta} - n\,\vec{\alpha}) - h(\vec{\beta}). \tag{7.6}$$

13

The definition (7.4) contains n-multiple integral over the contour on some Riemann surface [25]. Such a nontrivial closed contour exists and the integration is well defined, if the following condition is satisfied:

$$\vec{\beta} \cdot \vec{\alpha} = -\frac{1+m}{2} + \frac{1+n}{2}\, \vec{\alpha}^{\,2}\,, \quad m\,,n \in \mathbb{Z}\,. \tag{7.7}$$

The level (7.6), in which the null state appears is

$$N = \frac{n\,m}{2} \qquad (\vec{\alpha}^{\,2} \neq 0)\,, \tag{7.8}$$
$$N = \frac{|m|}{2} \qquad (\vec{\alpha}^{\,2} = 0)\,.$$

If there is a null state on the level N, the Kac determinant on this level vanishes. The vanishing curves of the Kac determinant are given in the table.

$0 =$	N(level)	NS	Ramond
$\beta_1 - \mathrm{i}\,\beta_2 + \frac{\sqrt{3+c}}{3}\frac{(1-l)}{2}$	$\frac{|l|}{2}$	$l \in 2\mathbb{Z}+1$	$l \in 2\mathbb{Z}$
$\beta_1 - \frac{\sqrt{3+c}}{3}\frac{(1-m)}{2} + \frac{3}{\sqrt{3+c}}\frac{(1-n)}{2}$	$\frac{n\,m}{2} > 0$	$m+n \in 2\mathbb{Z}$	$m+n \in 2\mathbb{Z}+1$
$\mathrm{i}\,\beta_2 - \frac{\sqrt{3+c}}{3}\,(1-j) + \frac{3}{\sqrt{3+c}}\frac{(1+k)}{2}$	$j\,k > 0$	$j, k \in \mathbb{Z}$	$j, k \in \mathbb{Z}$
$\beta_1 + \mathrm{i}\,\beta_2 - \frac{\sqrt{3+c}}{3}\frac{(3-l)}{2} + \frac{3}{\sqrt{3+c}}$	$\frac{|l|}{2}$	$l \in 2\mathbb{Z}+1$	$l \in 2\mathbb{Z}$

$$(7.9)$$

The first two vanishing curves are obtained by inserting $\vec{\alpha}_1$ and $\vec{\alpha}_2$ in (7.7). The third screening vector $\vec{\alpha}_3$ gives the same equation as $\vec{\alpha}_2$. As we have already mentioned $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are simple roots of $B(1,1)$, $\vec{\alpha}_3$ is a negative root, a pair of $\vec{\alpha}_2$. $B(1,1)$ Lie superalgebra has 5 positive roots, 2 even: $\vec{\alpha}_2, 2\,(\vec{\alpha}_1 + \vec{\alpha}_2)$ and 3 odd: $\vec{\alpha}_1, \vec{\alpha}_1 + \vec{\alpha}_2, \vec{\alpha}_1 + 2\,\vec{\alpha}_2$. The last two lines in (7.9) correspond to positive roots $\vec{\alpha}_1 + \vec{\alpha}_2$ and $\vec{\alpha}_1 + 2\,\vec{\alpha}_2$ respectively. They can be obtained by inserting $Q = Q_{\vec{\alpha}_1} Q_{\vec{\alpha}_2}$ and $Q = Q_{\vec{\alpha}_1} Q_{\vec{\alpha}_2}^2$ in (7.4). The negative roots give the same expressions as their positive counterparts. The fifth positive root $2\,(\vec{\alpha}_1 + \vec{\alpha}_2)$ corresponds to composite screening operator $Q = (Q_{\vec{\alpha}_1} Q_{\vec{\alpha}_2})^2$, which produces the same null states as $Q = Q_{\vec{\alpha}_1} Q_{\vec{\alpha}_2}$.

The last vanishing curve can be also obtained from the first one by applying the $\mathbb{Z}_2$ symmetry transformation (6.9). Moreover the full set of 4 vanishing curves is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (6.9, 6.10).

Using (6.4), (6.5), (6.7) and (6.8) one can easily express the Kac determinant in a, h parameterization. As a result we get three types of vanishing curves in the 3-dimensional space of c, a, h.

NS sector:

$$\boxed{\begin{array}{c}
f^{\mathrm{NS}}_{m,n} = a + \frac{1}{18\,(12+c)}\left(\left(3+c\right)\frac{1-m}{2} - 9\,\frac{1-n}{2}\right)\left(\left(3+c\right)\frac{1+m}{2} - 9\,\frac{1+n}{2}\right) \\
m,n > 0 \qquad m+n \in 2\mathbb{Z} \qquad N = \frac{m\,n}{2} \\
\hline
g^{\mathrm{NS}}_{j,k} = a\,(12+c) - (1+h)\,(3+c) + \\
\frac{1}{18}\left(\left(3+c\right)(1-j) - 9\,\frac{1-k}{2}\right)\left(\left(3+c\right)(1+j) - 9\,\frac{1+k}{2}\right) \\
j,k > 0 \qquad N = j\,k \\
\hline
d^{\mathrm{NS}}_l = h^2 - \frac{a}{18}(12+c)\,l^2 + \frac{h}{12}\left(3 - c + \frac{3+c}{3}\,l^2\right) + \\
\frac{1}{576}\left(1 - l^2\right)\left((3-c)^2 - \frac{(3+c)^2}{9}\,l^2\right) \\
l > 0 \qquad l \in 2\mathbb{Z} + 1 \qquad N = \frac{l}{2}
\end{array}} \qquad (7.10)$$

Ramond sector:

$$\boxed{\begin{array}{c}
f^{\mathrm{R}}_{m,n} = a + \frac{(6-c)^2}{72\,(12+c)} + \frac{3}{\sqrt{2\,(12+c)}}\,r\,\sqrt{a + \frac{(6-c)^2}{72\,(12+c)}} + \\
\frac{1}{72\,(12+c)}\left(9\,(1+n) - (3+c)\,m\right)\left(9\,(1-n) + (3+c)\,m\right) \\
m,n > 0 \qquad m+n \in 2\mathbb{Z} + 1 \qquad N = \frac{m\,n}{2} \\
\hline
g^{\mathrm{R}}_{j,k} = a\,(12+c) - h\,(3+c) + \frac{3\sqrt{12+c}}{\sqrt{2}}\,r\,\sqrt{a + \frac{(6-c)^2}{72\,(12+c)}} - \frac{12+7c}{8} + \\
\frac{1}{72}\left(2\,(3+c)(1-j) + 9\,(1-k)\right)\left(2\,(3+c)(1+j) + 9\,(1+k)\right) \\
j,k > 0 \qquad N = j\,k \\
\hline
d^{\mathrm{R}}_l = h^2 - \frac{\sqrt{12+c}}{6\sqrt{2}}\,l^2\,r\,\sqrt{a + \frac{(6-c)^2}{72\,(12+c)}} - \\
\frac{h}{36}\left(3\,c - (3+c)\,l^2\right) - \frac{a}{18}(12+c)l^2 + \\
\frac{1}{5184}\left(9\,c^2 - 2\,(234 - 15\,c + 5\,c^2)l^2 + (3+c)^2 l^4\right) \\
l > 0 \qquad l \in 2\mathbb{Z} \qquad N = \frac{l}{2} \\
d^{\mathrm{R}}_0 = h - c/24
\end{array}} \qquad (7.11)$$

where $r = \pm 1$ distinguish between $\big|\uparrow\big\rangle$ and $\big|\downarrow\big\rangle$ states of Ramond sector.

The Kac determinant on level N is (up to c–dependent factor)

$$\det M^{\mathrm{NS}}_N(c,a,h) = \prod_{\substack{1 \le m\,n \le 2\,N \\ m+n\ \mathrm{even}}} (f^{\mathrm{NS}}_{m,n})^{P_{\mathrm{NS}}(N - \frac{m\,n}{2})} \times$$
$$\prod_{1 \le j\,k \le N} (g^{\mathrm{NS}}_{j,k})^{P_{\mathrm{NS}}(N - j\,k)} \prod_{\substack{1 \le l \le 2\,N \\ l\ \mathrm{odd}}} (d^{\mathrm{NS}}_l)^{\widetilde{P}_{\mathrm{NS}}(N - \frac{l}{2}, l)} \qquad (7.12)$$

for NS sector. The counting of states is given by generating partition functions:

$$\sum_n P_{\mathrm{NS}}(n)\,x^n = \prod_{k=1}^{\infty}\left(\frac{1 + x^{k-1/2}}{1 - x^k}\right)^2,$$
$$\sum_n \widetilde{P}_{\mathrm{NS}}(n,l)\,x^n = \frac{1}{1 + x^{l/2}}\prod_{k=1}^{\infty}\left(\frac{1 + x^{k-1/2}}{1 - x^k}\right)^2. \qquad (7.13)$$

15

The Kac determinant of Ramond type representation:

$$\det M_N^{\mathrm{R}}(c, a, h) = \prod_{\substack{1 \le m\, n \le 2\,N \\ m+n\, \mathrm{odd}}} (f_{m,n}^{\mathrm{R}})^{P_{\mathrm{R}}(N - \frac{m\,n}{2})} \times$$

$$\prod_{1 \le j\, k \le N} (g_{j,k}^{\mathrm{R}})^{P_{\mathrm{R}}(N - j\, k)} \prod_{\substack{0 \le l \le 2\,N \\ l\, \mathrm{even}}} (d_l^{\mathrm{R}})^{\widetilde{P}_{\mathrm{R}}(N - \frac{l}{2}, l)} \,, \tag{7.14}$$

where the generating functions are:

$$\sum_n P_{\mathrm{R}}(n)\, x^n = \prod_{k=1}^{\infty} \left(\frac{1 + x^k}{1 - x^k} \right)^2 ,$$

$$\sum_n \widetilde{P}_{\mathrm{R}}(n, l)\, x^n = \frac{1}{1 + x^{l/2}} \prod_{k=1}^{\infty} \left(\frac{1 + x^k}{1 - x^k} \right)^2 . \tag{7.15}$$

Computer calculations of the Kac determinant on the first few levels verify the formulae (7.10–7.15).

8   Unitary representations

Now we are ready to discuss the unitarity restrictions on NS and Ramond representations. Unitary representations appear only at discrete set of values of central charge (1.1). The necessary condition for representation to be unitary is that its internal dimension a is included in the Kac table of dimensions of correspondent Virasoro unitary minimal model (3.4). And of course the external dimension b should not be negative.

Two dimensional Ramond representation can be unitary only if both A_0 eigenvalues $a_\uparrow$ and $a_\downarrow = a_\uparrow + \Delta a$ belong to the Kac table (3.4) and if both B_0 eigenvalues $b_\uparrow = h - a_\uparrow$ and $b_\downarrow = h - a_\downarrow$ are positive or zero.

The natural question arises: what value of $a + \Delta a$ do we get choosing a from the Kac table? It is simple exercise to check that

$$a_{m,n} + \Delta a(a_{m,n}, c_p^{(1)}) = a_{m,n\pm 1} \,, \tag{8.1}$$

where $\pm$ corresponds to two choices of Δa in (5.10). Analogously for the descending series:

$$a_{m,n} + \Delta a(a_{m,n}, c_p^{(2)}) = a_{m\pm 1,n} \,. \tag{8.2}$$

So, we see that the pair of Ramond internal dimensions in the $c_p^{(1)}$ ($c_p^{(2)}$) model is a pair of horizontal (vertical) neighboring dimensions from the correspondent Kac table.

Since G and U currents are in $\Phi_{1,2}$ representation of the Virasoro minimal model (Section 4), the property (8.1) is explained by the minimal model fusions of $\Phi_{1,2}$:

$$\Phi_{1,2} \times \Phi_{m,n} = \Phi_{m,n-1} + \Phi_{m,n+1} \,. \tag{8.3}$$

We illustrate it on two examples, one from ascending, one from descending series of models.

Example 1. $c = 12/5$

The simplest model is $c = 3/2$ model (3.5). It is explicitly solved by free field representation (5.3, 5.13). So we take as example the next model $c = 12/5$. Its internal Virasoro part is the Tricritical Ising model $(c_A = 7/10)$, the Kac table of which is:

$$\begin{pmatrix} 0 & \frac{7}{16} & \frac{3}{2} \\ \frac{1}{10} & \frac{3}{80} & \frac{3}{5} \\ \frac{3}{5} & \frac{3}{80} & \frac{1}{10} \\ \frac{3}{2} & \frac{7}{16} & 0 \end{pmatrix}. \tag{8.4}$$

In the NS sector the representation $\big|\,a\,,b\,\big\rangle$ can be unitary only for $b \geq 0$ and a from the set

$$0 \quad \frac{3}{80} \quad \frac{1}{10} \quad \frac{7}{16} \quad \frac{3}{5} \quad \frac{3}{2} \,. \tag{8.5}$$

Ramond ground states $(a + b = \frac{1}{10})$:

$$\Big|\frac{1}{10},0\,\Big\rangle \qquad \Big|\frac{3}{80},\frac{1}{16}\Big\rangle \qquad \Big|0,\frac{1}{10}\Big\rangle. \tag{8.6}$$

For two dimensional Ramond states the pairs of internal dimensions $\left\{\begin{matrix} a_\uparrow \\ a_\downarrow \end{matrix}\right\}$ are:

$$\left\{\begin{matrix} 0 \\ \frac{7}{16} \end{matrix}\right\} \quad \left\{\begin{matrix} \frac{7}{16} \\ \frac{3}{2} \end{matrix}\right\} \quad \left\{\begin{matrix} \frac{1}{10} \\ \frac{3}{80} \end{matrix}\right\} \quad \left\{\begin{matrix} \frac{3}{80} \\ \frac{3}{5} \end{matrix}\right\}. \tag{8.7}$$

Example 2. $c = 12$

The simplest example from descending series is $c = 12$ model. Its internal subalgebra is the Ising model $(c_A = 1/2)$. This example is exactly the case of $Spin(7)$ algebra discussed in [8]. The Kac table of the Ising model:

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{16} & \frac{1}{16} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{8.8}$$

17

So, the unitary internal dimensions are

$$0 \quad \frac{1}{16} \quad \frac{1}{2}. \tag{8.9}$$

Ramond ground states ($h = c/24 = 1/2$):

$$\left| \frac{1}{2}, 0 \right\rangle \qquad \left| \frac{1}{16}, \frac{7}{16} \right\rangle \qquad \left| 0, \frac{1}{2} \right\rangle. \tag{8.10}$$

The pairs of Ramond A_0 eigenvalues:

$$\left\{ \begin{matrix} 0 \\ \frac{1}{16} \end{matrix} \right\} \quad \left\{ \begin{matrix} \frac{1}{16} \\ \frac{1}{2} \end{matrix} \right\}. \tag{8.11}$$

The highest weight representation of $\mathcal{SW}(3/2, 2)$ can be decomposed to a direct sum of representations of its internal subalgebra. The weights of all representations in the decomposition should also belong to the unitary set of dimensions (3.4). We have to study the descendants of highest weight state of $\mathcal{SW}(3/2, 2)$ algebra, which are highest weight representations with respect to internal subalgebra A. The simplest examples of such states are the level $\frac{1}{2}$ descendants of NS highest weight state. They are annihilated by positive A modes:

$$A_n \, G_{-\frac{1}{2}} \left| a, b \right\rangle = A_n \, U_{-\frac{1}{2}} \left| a, b \right\rangle = 0 \qquad \forall n > 0. \tag{8.12}$$

However these states are not A_0 eigenstates. But one can construct two linear combinations of these states, which are A_0 eigenstates and therefore highest weight representations of the internal Virasoro algebra:

$$\left| \chi_{1,2} \right\rangle = \left(\frac{12 + c \pm 3 \sqrt{(6-c)^2 + 72 \, a \, (12+c)}}{2 \sqrt{15-c} \sqrt{21+4\,c}} \, G_{-1/2} + U_{-1/2} \right) \left| a, b \right\rangle,$$
$$A_0 \left| \chi_{1,2} \right\rangle = \widetilde{a}_{1,2} \left| \chi_{1,2} \right\rangle, \tag{8.13}$$

where $\widetilde{a}$ is a function of a, b, c. Providing that a is one of the Kac dimensions $\widetilde{a}$ has also to be included in the Kac table. B_0 eigenvalue $\widetilde{b} = h + \frac{1}{2} - \widetilde{a}$ should be nonnegative. $\Delta a_{1,2} = \widetilde{a}_{1,2} - a$ is the same as the difference of Ramond internal dimensions (5.10).

For the case of interest, $c = c_p^{(1)}$ ($c = c_p^{(2)}$) the $\left| a_{m,n}, b \right\rangle$ state has two level $\frac{1}{2}$ descendants, which are A primaries with internal dimensions being nearest to $a_{m,n}$ neighbors in the Kac table: $\widetilde{a} = a_{m,n+1}$ and $\widetilde{a} = a_{m,n-1}$ ($a_{m+1,n}$ and $a_{m-1,n}$), where we define $a_{m,0} = a_{m+p,p-1}$ using periodicity of Kac dimensions. Again the shift in the Kac table is explained by fusion (8.3).

18

We have to take into account another possibility that one of the states $\left| \chi_{1,2} \right\rangle$ is null and should be modded out to make the representation irreducible. In such a case there is only one descendant on level $\frac{1}{2}$. It happens when the values of a, b, c lie on the zero locus of the Kac determinant on level $\frac{1}{2}$: $f_{1,1} = 0$ (it's essentially $a = 0$ for every c) and $d_1 = 0$.

At $c = c_p^{(1)}$ ($c = c_p^{(2)}$) the states $\left| a_{m,1}, b \right\rangle$ and $\left| a_{m,p-1}, b \right\rangle$ ($\left| a_{1,n}, b \right\rangle$ and $\left| a_{p,n}, b \right\rangle$) from the first and last columns (rows) of the correspondent Kac table generically are not unitary, since they have one of the descendants, that is "neighbor" from outside of the Kac table. "Generically" is in the meaning that this is not true for intersections with $f_{1,1} = 0$ and $d_1 = 0$, where the nonunitary descendant becomes null. The hall line $\{a = a_{1,1} = 0, b \geq 0\}$ should not be considered as nonunitary, because it coincides with the line $f_{1,1} = 0$.

Example 1. $c = 12/5$

All the region below $d_1 = 0$ curve (Figure 1) is nonunitary, since The Kac determinant on level $\frac{1}{2}$ is negative. $\left| \frac{1}{10}, b \right\rangle$ has level $\frac{1}{2}$ descendants with A_0 weights $\widetilde{a}_1 = a_{2,0} = a_{3,4} = \frac{63}{80}$, $\widetilde{a}_2 = a_{2,2} = \frac{3}{80}$; the descendants of $\left| \frac{3}{5}, b \right\rangle$: $\widetilde{a}_1 = a_{3,0} = a_{2,4} = \frac{143}{80}$, $\widetilde{a}_2 = a_{3,2} = \frac{3}{80}$; the descendants of $\left| \frac{3}{2}, b \right\rangle$: $\widetilde{a}_1 = a_{4,0} = a_{1,4} = \frac{51}{16}$, $\widetilde{a}_2 = a_{1,2} = \frac{7}{16}$. In all three cases $\widetilde{a}_1$ is not from the unitary set, therefore $\left| \frac{1}{10}, b \right\rangle$, $\left| \frac{3}{5}, b \right\rangle$ and $\left| \frac{3}{2}, b \right\rangle$ are not unitary except the case then $\left| \chi_1 \right\rangle$ is null. It happens only on $d_1 = 0$ curve: $\left| \frac{1}{10}, \frac{1}{10} \right\rangle$ and $\left| \frac{3}{5}, 0 \right\rangle$. $a = \frac{3}{2}$ doesn't intersect $d_1 = 0$, therefore $\left| \frac{3}{2}, b \right\rangle$ are all nonunitary. The nonunitary descendant of $\left| 0, b \right\rangle$ is null for every b, the state stays unitary.

States with weights $a = \frac{7}{16}$ and $a = \frac{3}{80}$ are from the middle of the Kac table, $\left| \frac{7}{16}, b \right\rangle$ has descendants on level $\frac{1}{2}$ with $\widetilde{a}_1 = 0$ and $\widetilde{a}_2 = \frac{3}{2}$, $\left| \frac{3}{80}, b \right\rangle$ has descendants with $\widetilde{a}_1 = \frac{1}{10}$ and $\widetilde{a}_2 = \frac{3}{5}$, so there is no restriction on unitarity of $\left| \frac{7}{16}, b \geq \frac{1}{2} \right\rangle$ and $\left| \frac{3}{80}, b \geq \frac{1}{16} \right\rangle$.

Example 2. $c = 12$

Again the region below $d_1 = 0$ is nonunitary (Figure 2). $\left| \frac{1}{2}, \frac{1}{2} \right\rangle$ is the only unitary state on $a = \frac{1}{2}$ line. $\left| \frac{1}{16}, b \geq \frac{7}{16} \right\rangle$ has descendants with $\widetilde{a} = 0$ and $\widetilde{a} = \frac{1}{2}$, it remains unitary. $\left| 0, b \geq 0 \right\rangle$ is unitary since it lies on $f_{1,1} = 0$ line.

Now we proceed to the deeper levels. Two level 1 descendants

$$
\begin{aligned}
G_{-1/2}\, U_{-1/2} - \frac{54\,h + 3 - 2\,c}{\sqrt{15 - c}\,\sqrt{21 + 4\,c}}\, A_{-1} \left| a, b \right\rangle, \\
B_{-1} \left| a, b \right\rangle
\end{aligned}
\tag{8.14}
$$

are A–primaries having the same A_0 eigenvalue a as $\left| a, b \right\rangle$.

Figure 1: NS sector of $c = 12/5$ model. Thick lines and dots show the continuous and discrete unitary spectrum respectively, thin lines show the vanishing locus of the Kac determinant.
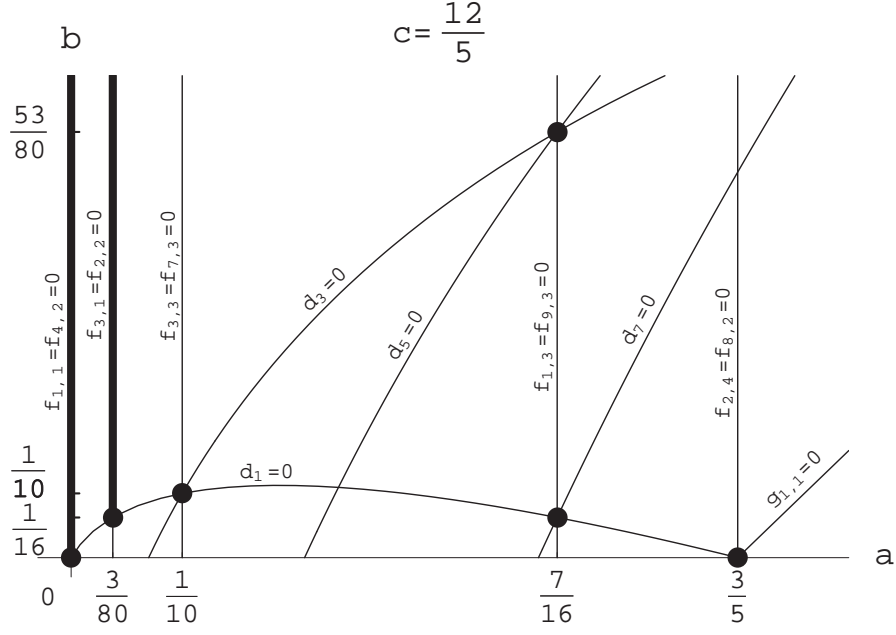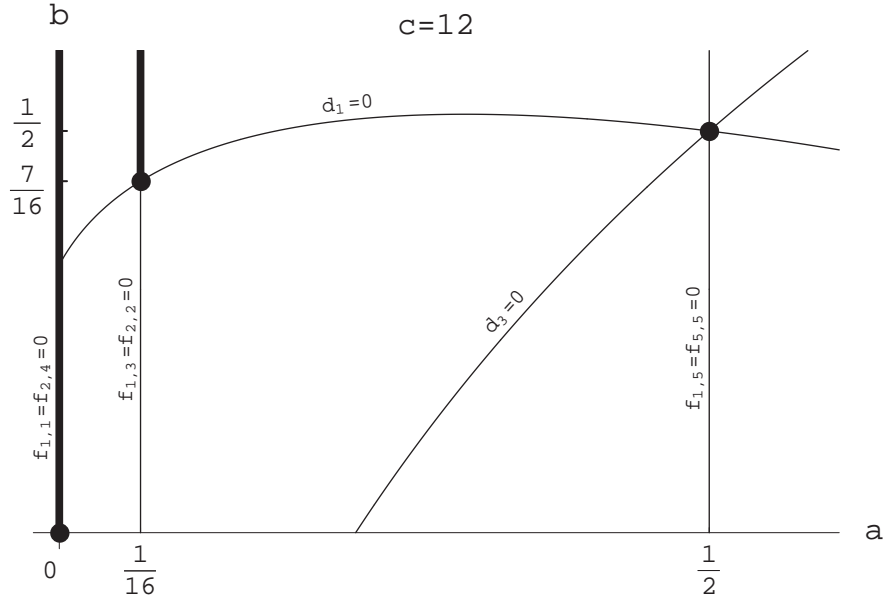

Figure 2: NS sector of $c = 12$ model. Thick lines and dots show the continuous and discrete unitary spectrum respectively, thin lines show the vanishing locus of the Kac determinant.

Level $\frac{3}{2}$ descendants repeat the situation on level $\frac{1}{2}$: they can be organized to A highest weight representations with A_0 eigenvalue being nearest to a neighbor in the row (column) of the Kac table.

Descendants with a new feature appear on level 2. Starting from this level one can act twice by the modes of G: $G_{-3/2} G_{-1/2} | a_{m,n}, b \rangle$, producing shift by 2 in the Kac table by repeated action of $\Phi_{1,2}$ (8.3). A_0 eigenvalues are next to nearest neighbors to $a_{m,n}$ in the same row: $a_{m,n\pm2}$ (for models in ascending series), or in the same column: $a_{m\pm2,n}$ (for models in descending series) of the correspondent Kac table. Appearance of these descendants restricts the unitarity of highest weight NS representation $| a, b \rangle$, if its internal dimension a belongs to second or last but one column (row) of the Kac table, since at least one of its new descendants is from outside of the Kac table. Again there is one possibility to save the unitarity: to find such a value of b, that the nonunitary descendant of $| a, b \rangle$ becomes a null state. This can happen only on vanishing locus of $\det M_2^{\mathrm{NS}}$, $\mathcal{SW}(3/2, 2)$ Kac determinant on level 2.

As we go to the deeper levels, we get a new shift in the Kac table on level $\frac{9}{2}$ (the level of $G_{-5/2} G_{-3/2} G_{-1/2} | a, b \rangle$). The descendants of $| a_{m,n}, b \rangle$ have A_0 eigenvalues $a_{m,n\pm3}$ ($a_{m\pm3,n}$). One can proceed to build new descendants which are A primaries those A_0 eigenvalue shifted farther in the Kac table. Every highest weight representation $| a_{m,n}, b \rangle$ gets nonunitary descendant on large enough level. To preserve its unitarity the representation have to be degenerated on this level in a way, that the nonunitary descendant is a null state.

In general, the shift by k in the Kac table appears starting from level

$$\sum_{j=1}^{k} \left(j - \frac{1}{2} \right) = \frac{1}{2} k^2. \tag{8.15}$$

Example 1. $c = 12/5$

$| \frac{3}{80}, b \rangle$ would have two nonunitary descendants on level 2: $\widetilde{a} = \frac{63}{80}$ and $\widetilde{a} = \frac{143}{80}$, but both are null, since $a = \frac{3}{80}$ lies on $f_{3,1} = f_{2,2} = 0$, meaning that there are independent null states on levels $N = \frac{3}{2}$ and $N = 2$.

Level 2 nonunitary descendants of $| \frac{7}{16}, b \rangle$: $\widetilde{a} = \frac{3}{16}$ and $\widetilde{a} = \frac{51}{16}$. One of them is null, since $| \frac{7}{16}, b \rangle$ lies on $f_{1,3} = 0$ $\left(N = \frac{3}{2} \right)$, the second is null only on intersection with $d_1 = 0$ or $d_3 = 0$: $| \frac{7}{16}, \frac{1}{16} \rangle$ and $| \frac{7}{16}, \frac{53}{80} \rangle$ respectively.

The full analysis shows that there are no other restrictions on unitarity from deeper than $\frac{1}{2}$ levels. As a final result we get a picture of unitary representations at $c = 12/5$ (Figure 1).

Example 2. $c = 12$

No new unitarity restrictions from deeper than $\frac{1}{2}$ levels. Figure 2 represents the unitary spectrum of the model.

The structure of Ramond representations is similar to that of NS representations. The descendants are shifted in the row (column) of the Kac table. The nearest neighbors to the pair of Ramond internal dimensions appear at level 1, next to nearest – at level 3. Shift by k starts from level

$$\sum_{j=1}^{k} j = \frac{1}{2} k (k+1). \tag{8.16}$$

We have shown, that every unitary state of $\mathcal{SW}(3/2, 2)$ algebra is decomposed to a sum of representations of internal Virasoro algebra A with internal dimensions belonging to the same row (column) of the Kac table. Moreover, *every* dimension in the row (column) is present in the decomposition of the unitary state. In tables 1 and 2 we list unitary states in NS and Ramond sectors of our two example models and the decomposition of the states to highest weight representations of the internal algebra. The new notation is introduced: $\{a, b\}$ stays for highest weight state of A Virasoro algebra, which is eigenstate of B_0 operator:

$$\begin{aligned} A_n \{a, b\} &= 0 \qquad n \geq 0, \\ A_0 \{a, b\} &= a \{a, b\}, \\ B_0 \{a, b\} &= b \{a, b\}. \end{aligned} \tag{8.17}$$

When we write $\{a, b\}$ we actually mean infinite number of states $\{a, b+k\}$, k being positive integer or zero. These states are obtained by action of negative modes of B on $\{a, b\}$. Obviously, $\{a, b+k\}$ also satisfies definition (8.17) with B_0 eigenvalue $b + k$.

The lengthly list of all the unitary representations of descending and ascending series of models is given in appendix B. We would like to point out here some common features of the models.

- At fixed c_p there are unitary lines (continuous spectrum) and separated points (discrete spectrum) in two dimensional space of weights. We include the points of beginning of unitary lines to discrete spectrum. Continuous spectrum of each sector consists of $[\frac{p(p-2)+1}{4}]$ lines at $c = c_p^{(1)}$ and of $[\frac{(p+1)(p-1)+1}{4}]$ lines at $c = c_p^{(2)}$. Discrete spectrum consists of $\frac{p(p+1)}{2}$ states in each sector at $c = c_p^{(1)}$ or $c = c_p^{(2)}$.

- Every model contains in its discrete spectrum NS state of dimension $h = \frac{1}{2}$: $\left| a_{1,2}, \frac{1}{2} - a_{1,2} \right\rangle$ in ascending series and $\left| a_{2,1}, \frac{1}{2} - a_{2,1} \right\rangle$ in descending series. In

22

NS			Ramond		
state	decomposition		state	decomposition	
$\left|0,0\right\rangle$	$\left\{\frac{7}{16},\frac{17}{16}\right\}$	$\left\{\frac{3}{2},\frac{5}{2}\right\}$	$\left|\begin{smallmatrix}\frac{7}{16},\frac{17}{16}\\\frac{3}{2},0\end{smallmatrix}\right\rangle$	$\left\{0,\frac{5}{2}\right\}$	
$\left|\frac{7}{16},\frac{1}{16}\right\rangle$	$\{0,1\}$	$\left\{\frac{3}{2},\frac{1}{2}\right\}$	$\left|\begin{smallmatrix}0,\frac{1}{2}\\\frac{7}{16},\frac{1}{16}\end{smallmatrix}\right\rangle$	$\left\{\frac{3}{2},1\right\}$	
$\left|\frac{7}{16},\frac{53}{80}\right\rangle$	$\left\{0,\frac{8}{5}\right\}$	$\left\{\frac{3}{2},\frac{1}{10}\right\}$	$\left|0,\frac{1}{10}\right\rangle$	$\left\{\frac{7}{16},\frac{53}{80}\right\}$	$\left\{\frac{3}{2},\frac{8}{5}\right\}$
$\left|\frac{1}{10},\frac{1}{10}\right\rangle$	$\left\{\frac{3}{80},\frac{53}{80}\right\}$	$\left\{\frac{3}{5},\frac{8}{5}\right\}$	$\left|\begin{smallmatrix}\frac{3}{80},\frac{53}{80}\\\frac{3}{5},\frac{1}{10}\end{smallmatrix}\right\rangle$	$\left\{\frac{1}{10},\frac{8}{5}\right\}$	
$\left|\frac{3}{80},\frac{1}{16}\right\rangle$	$\left\{\frac{1}{10},\frac{1}{2}\right\}$	$\left\{\frac{3}{5},1\right\}$	$\left|\frac{3}{80},\frac{1}{16}\right\rangle$	$\left\{\frac{1}{10},1\right\}$	$\left\{\frac{3}{5},\frac{1}{2}\right\}$
$\left|\frac{3}{5},0\right\rangle$	$\left\{\frac{3}{80},\frac{17}{16}\right\}$	$\left\{\frac{1}{10},\frac{5}{2}\right\}$	$\left|\frac{1}{10},0\right\rangle$	$\left\{\frac{3}{80},\frac{17}{16}\right\}$	$\left\{\frac{3}{5},\frac{5}{2}\right\}$
$\left|0,x\right\rangle$	$\left\{\frac{7}{16},\frac{1}{16}+x\right\}$	$\left\{\frac{3}{2},\frac{1}{2}+x\right\}$	$\left|\begin{smallmatrix}0,\frac{1}{2}+x\\\frac{7}{16},\frac{1}{16}+x\end{smallmatrix}\right\rangle$	$\left\{\frac{3}{2},x\right\}$	
$\left|\frac{3}{80},\frac{1}{16}+x\right\rangle$	$\left\{\frac{1}{10},\frac{1}{2}+x\right\}$	$\left\{\frac{3}{5},x\right\}$	$\left|\begin{smallmatrix}\frac{1}{10},x\\\frac{3}{80},\frac{1}{16}+x\end{smallmatrix}\right\rangle$	$\left\{\frac{3}{5},\frac{1}{2}+x\right\}$	

Table 1: Unitary states of $c = 12/5$ model and their decomposition.

NS			Ramond		
state	decomposition		state	decomposition	
$\left|0,0\right\rangle$	$\left\{\frac{1}{16},\frac{23}{16}\right\}$	$\left\{\frac{1}{2},\frac{7}{2}\right\}$	$\left|\frac{1}{2},0\right\rangle$	$\left\{\frac{1}{16},\frac{23}{16}\right\}$	$\left\{0,\frac{7}{2}\right\}$
$\left|\frac{1}{16},\frac{7}{16}\right\rangle$	$\{0,1\}$	$\left\{\frac{1}{2},\frac{3}{2}\right\}$	$\left|\frac{1}{16},\frac{7}{16}\right\rangle$	$\left\{0,\frac{3}{2}\right\}$	$\left\{\frac{1}{2},1\right\}$
$\left|\frac{1}{2},\frac{1}{2}\right\rangle$	$\left\{\frac{1}{16},\frac{23}{16}\right\}$	$\{0,3\}$	$\left|0,\frac{1}{2}\right\rangle$	$\left\{\frac{1}{16},\frac{23}{16}\right\}$	$\left\{\frac{1}{2},3\right\}$
$\left|0,x\right\rangle$	$\left\{\frac{1}{16},\frac{7}{16}+x\right\}$	$\left\{\frac{1}{2},\frac{3}{2}+x\right\}$	$\left|\begin{smallmatrix}\frac{1}{2},x\\\frac{1}{16},\frac{7}{16}+x\end{smallmatrix}\right\rangle$	$\left\{0,\frac{3}{2}+x\right\}$	
$\left|\frac{1}{16},\frac{7}{16}+x\right\rangle$	$\{0,1+x\}$	$\left\{\frac{1}{2},\frac{1}{2}+x\right\}$	$\left|\begin{smallmatrix}\frac{1}{16},\frac{7}{16}+x\\0,\frac{1}{2}+x\end{smallmatrix}\right\rangle$	$\left\{\frac{1}{2},1+x\right\}$	

Table 2: Unitary states of $c = 12$ model and their decomposition.

Section 10 we extend the $\mathcal{SW}(3/2, 2)$ algebra to include this state. In addition every model in descending series contains NS state of dimension $h = 1$: $\left| a_{3,1}, \frac{1}{2} - a_{3,1} \right\rangle$. All the models include NS continuous line: $\left| 0, b \geq 0 \right\rangle$.

- The purely internal states, i.e. having zero external dimension $b = 0$, are important for analysis of fusion rules (see the next section). In addition to the vacuum state $\left| 0, 0 \right\rangle$ there is always $\left| a_{3,1}, 0 \right\rangle$ $(\left| a_{1,3}, 0 \right\rangle)$ state in the NS discrete spectrum. Ramond sector contains ground state $\left| a_{2,1}, 0 \right\rangle$ $(\left| a_{1,2}, 0 \right\rangle)$ and two dimensional state $\left| \begin{matrix} a_{4,2}, a_{4,1} - a_{4,2} \\ a_{4,1}, \quad 0 \end{matrix} \right\rangle$ $\left(\left| \begin{matrix} a_{2,4}, a_{1,4} - a_{2,4} \\ a_{1,4}, \quad 0 \end{matrix} \right\rangle \right)$. The existence of these representations was predicted in the end of Section 4 using relation (4.10). There were other local or semilocal states $\{a_{m,1}, 0\}$ $(\{a_{1,n}, 0\})$, but they have higher than 1 or $\frac{3}{2}$ poles in their OPE with G, therefore they are secondary fields. The correspondent primary fields are easily found in the discrete spectrum. For example, in the $c = 3$ $(p = 5)$ model the NS state $\{a_{5,1} = 3, 0\}$ is level $\frac{1}{2}$ descendant of $\left| \frac{7}{5}, \frac{11}{10} \right\rangle$. In the $c = 24/7$ $(p = 6)$ model $\left\{ a_{5,1} = \frac{22}{7}, 0 \right\}$ is level $\frac{1}{2}$ descendant of $\left| \frac{85}{56}, \frac{9}{8} \right\rangle$ and $\{a_{6,1} = 5, 0\}$ is level 1 descendant of $\left| \begin{matrix} \frac{4}{3}, \frac{8}{3} \\ \frac{23}{8}, \frac{9}{8} \end{matrix} \right\rangle$.

- There is the same number of states in NS and Ramond sectors, the number of discrete states is the same and the number of continuous lines is the same. We discuss it in the next section.

9 Fusion rules and minimal models

We can not say exactly what are the fusion rules of all fields in the unitary model. However there are some selection rules induced by the fusion rules of internal Virasoro minimal model [6]:

$$\Phi_{m_1, n_1} \times \Phi_{m_2, n_2} = \sum_{m = |m_1 - m_2| + 1}^{m_1 + m_2 - 1} \sum_{n = |n_1 - n_2| + 1}^{n_1 + n_2 - 1} \Phi_{m,n}, \qquad (9.1)$$

where the indices m and n in the sums are raised by steps of 2. Essentially these fusion rules are two separate selection rules: one is for the row number, another is for the column number. In the case of ascending (descending) series model the column (row) selection rule has no meaning any more, since any unitary $\mathcal{SW}(3/2, 2)$ representation contains a whole row (column) of the minimal model Kac table. But another selection rule, row (column) rule is still valid. In $c = c_p^{(1)}$ model the Kac table row number of the internal dimension satisfies the $SU(2)$ selection rule:

$$m \in \{|m_1 - m_2| + 1, |m_1 - m_2| + 3, \ldots, |m_1 + m_2| - 1\}. \qquad (9.2)$$

(The same rule for column number n in $c = c_p^{(2)}$ model.) Moreover every row in the set must be represented in the fusion rule.

We have mentioned in the previous section, that every model contains a field with zero external dimension. Fusion rules of such a field can be written exactly, since it doesn't change the external dimension of fields, it is acting on. Take for example the fusion of $\left| \frac{3}{5} , 0 \right\rangle$ state of $c = 12/5$ model with another state from the unitary spectrum, e.g. $\left| \frac{1}{10} , \frac{1}{10} \right\rangle$:

$$\left| \frac{3}{5} , 0 \right\rangle \times \left| \frac{1}{10} , \frac{1}{10} \right\rangle = \left\{ \frac{1}{10} , \frac{1}{10} \right\} + \left\{ \frac{3}{2} , \frac{1}{10} \right\} , \tag{9.3}$$

where internal dimensions are obtained according to the fusion rules of the Tricritical Ising model. $\left\{ \frac{1}{10} , \frac{1}{10} \right\}$ is of course simply the $\left| \frac{1}{10} , \frac{1}{10} \right\rangle$ state, and $\left\{ \frac{3}{2} , \frac{1}{10} \right\}$ is found to be the level $\frac{1}{2}$ descendant of $\left| \frac{7}{16} , \frac{53}{80} \right\rangle$ (see table 1). Another example from the same model:

$$\left| \begin{matrix} \frac{7}{16} , \frac{17}{16} \\ \frac{3}{2} , 0 \end{matrix} \right\rangle \times \left| \begin{matrix} \frac{1}{10} , \\ \frac{3}{80} , \frac{1}{16} + x \end{matrix} x \right\rangle = \left\{ \frac{3}{5} , x \right\} + \left\{ \frac{3}{80} , \frac{1}{16} + x \right\} = \left| \frac{3}{80} , \frac{1}{16} + x \right\rangle . \tag{9.4}$$

Some states have purely internal descendants, as was shown in the end of the previous section. Due to this feature we also know all the fusions of these representations. For example, in the $c = 3$ model the fusion of $\left| \frac{7}{5} , \frac{11}{10} \right\rangle$ with any other field is dictated by its $\{ 3 , 0 \}$ descendant. In particular, its square is identity:

$$\left| \frac{7}{5} , \frac{11}{10} \right\rangle \times \left| \frac{7}{5} , \frac{11}{10} \right\rangle = \left| 0 , 0 \right\rangle . \tag{9.5}$$

Every $\mathcal{SW}(3/2, 2)$ model contains such a representation in its discrete spectrum, since every Virasoro minimal model contains $\Phi_{p,1} = \Phi_{1,p-1}$ field, square of which is identity. From relation (4.10) we get that such a representation is of NS type for p odd and of Ramond type for p even in the ascending series of models and vice versa in the descending series of models. This is closely related to the fact, that there is the same number of states in different sectors.

$N = 1$ and $N = 2$ superconformal minimal models also have equal number of states in NS and Ramond sectors. In the $N = 2$ case ($c = \frac{3k}{k+2}$, $k = 1, 2, \dots$) there is an isomorphism between two sectors preserving the fusion rule structure. Moreover there exists a continuous transformation, $U(1)$ flow, connecting the isomorphic states [26]. In the case of $N = 1$ superconformal minimal models ($c = \frac{3}{2} - \frac{12}{k(k+2)}$, $k = 3, 4, \dots$) there is no such a continuous flow, but in every second model ($k = 3, 5, 7, \dots$) the isomorphism exists. The isomorphism map is performed by Ramond state, fusion of which with itself gives the vacuum ($h = \frac{7}{16}$ Ramond state in the Tricritical Ising model).

In the case of $\mathcal{SW}(3/2, 2)$ algebra the situation is the same as for $N = 1$ superconformal algebra. $\mathcal{SW}(3/2, 2)$ algebra admits no continuous transformation between the sectors. The isomorphism exists in every second model, exactly then the state $\{a_{p,1}, 0\}$ ($\{a_{1,p-1}, 0\}$) is of Ramond type. It serves as the isomorphism map. One can find two examples of the NS – Ramond correspondence in Tables 1 and 2: the representations in the same row are isomorphic.

Finally we come to the question, what are the minimal models of $\mathcal{SW}(3/2, 2)$ algebra? The states of continuous spectrum can not be included in the minimal models. Their fusions are not defined, since the continuous spectrum representations are not completely degenerated, they have only 2 independent null vectors. The fusions of discrete spectrum representations in general also can not be defined, they depend on the particular realization of the algebra and may include representations from continuous spectrum. For example, in the $c = 3/2$ model (3.5, 5.3, 5.13) one can compactify the free boson on any radius. Then the fusion of the boson twist field s with itself depends on the compactification radius.

However, in every model there is a subset of discrete spectrum fields with well defined fusion rules, and the set is closed under these fusion rules. It consists of the purely internal representations and the representations having purely internal descendants: $\{a_{m,1}, 0\}$ ($\{a_{1,n}, 0\}$). This is in some sense trivial set of fields, but the only one, guaranteed to close under fusion rules. So, the $\mathcal{SW}(3/2, 2)$ minimal models in addition to the ground states

$$\text{NS: } a = h = 0, \qquad \text{R: } a = h = c/24 \tag{9.6}$$

contain at $c = 6 - \dfrac{18}{p+1}$ (ascending series of minimal models) $p - 2$ fields

$$
\begin{aligned}
a &= \frac{(p\,(m+3) - (m+1) - r\,(p+1))^2 - 4}{16\,p\,(p+1)}, \\
h &= \frac{m^2\,(p-1) + 2\,m\,(3\,p-1) + p - 5}{8\,(p+1)} + \frac{|r|}{8}, \\
m &= 1, 2, \ldots, p-2, \qquad \text{NS: } m - \text{odd}, r = 0, \quad \text{R: } m - \text{even}, r = \pm 1
\end{aligned}
\tag{9.7}
$$

and at $c = 6 + \dfrac{18}{p}$ (descending series of minimal models) $p - 3$ fields

$$
\begin{aligned}
a &= \frac{(p\,(n+3) + 2\,n + 4 - r\,p)^2 - 4}{16\,p\,(p+1)}, \\
h &= \frac{n^2\,(p+2) + 2\,n\,(3\,p+4) + p + 6}{8\,p} + \frac{|r|}{8}, \\
n &= 1, 2, \ldots, p-3, \qquad \text{NS: } n - \text{odd}, r = 0, \quad \text{R: } n - \text{even}, r = \pm 1.
\end{aligned}
\tag{9.8}
$$

26

One can get the minimal model representations (excluding the vacuum state) as intersections of $g_{1,1} = 0$ and $d_m = 0$ vanishing curves of the Kac determinant.

There are also nonunitary minimal models of $\mathcal{SW}(3/2, 2)$ algebra. They correspond to nonunitary minimal models of internal Virasoro algebra, central charge of which is

$$c_A = 1 - \frac{6\,(p-q)^2}{p\,q}\,. \tag{9.9}$$

Then the full central charge is

$$c = -6 + 18\,\frac{p-q}{p}\,. \tag{9.10}$$

The condition of unitarity is $|p - q| \leq 1$. We do not study the representation theory of nonunitary minimal models here.

10 Extensions of $\mathcal{SW}(3/2, 2)$ algebra

In this section we discuss two types of extensions of $\mathcal{SW}(3/2, 2)$ algebra.

10.1 $\mathcal{SW}(3/2, 3/2, 2)$ algebra

We have mentioned in the end of Section 8 that every unitary model contains dimension $\frac{1}{2}$ field in its discrete spectrum. It lies on intersection of $f_{1,3}^{\mathrm{NS}} = 0$ and $d_1^{\mathrm{NS}} = 0$. For general value of central charge the intersection is $\left| \frac{15-c}{2\,(12+c)}, \frac{1}{2} - \frac{15-c}{2\,(12+c)} \right\rangle$. We denote the field corresponding to this state by Ψ, and the correspondent $\mathcal{SW}(3/2, 2)$ conformal family – by $[\Psi]$.

We are going to show, that $[\Psi]$ multiplet consists of 4 Virasoro primary fields Ψ, Φ, Ω and Σ of dimensions $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2 respectively; and that $\mathcal{SW}(3/2, 2)$ algebra can be uniquely extended to include this multiplet. The method, we use, is similar to that of Section 4 and we omit here detailed calculations.

Field $\Psi(w)$ obeys OPEs (5.2) with $h = \frac{1}{2}$ and $a = \frac{15-c}{2\,(12+c)}$. $(G_{-1/2}\Psi)$ and $(U_{-1/2}\Psi)$ are not independent due to null vector on level $\frac{1}{2}$:

$$\left(U_{-1/2} - \frac{2\sqrt{15-c}}{\sqrt{21+4\,c}}\,G_{-1/2} \right) \Psi = 0\,. \tag{10.1}$$

One new field of dimension 1 has to be introduced:

$$\Phi = G_{-1/2}\Psi\,. \tag{10.2}$$

The field

$$\Omega = A_{-1}\Psi - \frac{15-c}{12+c}\partial\Psi \tag{10.3}$$

is T–primary of dimension $\frac{3}{2}$.

The second independent null vector appears on level $\frac{3}{2}$:

$$A_{-1}G_{-1/2}\Psi = A_{-1}U_{-1/2}\Psi = 0. \tag{10.4}$$

So the only new field of dimension 2 is $U_{-3/2}\Psi$. We define

$$\Sigma = \frac{\sqrt{15-c}\,\sqrt{21+4c}}{3\,(12+c)}U_{-3/2}\Psi - \frac{15-c}{12+c}L_{-1}G_{-1/2}\Psi. \tag{10.5}$$

The fields $A_{-1}^2\Psi$, $U_{-3/2}G_{-1/2}\Psi$, $A_{-1}U_{-3/2}\Psi$ enter to the OPEs of $\mathcal{SW}(3/2,2)$ generators with Ψ, Φ, Ω, Σ. But due to descendants of the two null vectors discussed above they can be substituted by composites and derivatives of the earlier introduced fields:

$$A_{-1}^2\Psi = \frac{18}{12+c}A_{-2}\Psi,$$

$$U_{-3/2}G_{-1/2}\Psi = -\frac{3\,(12+c)}{\sqrt{15-c}\,\sqrt{21+4c}}L_{-1}A_{-1}\Psi, \tag{10.6}$$

$$A_{-1}U_{-3/2}\Psi = -\frac{27}{\sqrt{15-c}\,\sqrt{21+4c}}A_{-1}G_{-3/2}\Psi.$$

Using the definitions and the null vectors above and $\mathcal{SW}(3/2,2)$ commutation relations one can write all the OPEs of type $X(z)Y(w)$, where X is one of $\mathcal{SW}(3/2,2)$ generators $\{T, A, G, U\}$, Y is one of 4 fields, which constitute the $[\Psi]$ multiplet: $\{\Psi, \Phi, \Omega, \Sigma\}$. We do not write these OPEs here because of their length. One thing to mention: (Ψ, Φ) and (Ω, Σ) form $h = \frac{1}{2}$ and $h = \frac{3}{2}$ multiplets of $N = 1$ superconformal algebra generated by T and G.

Now one need to close the extended algebra. The OPEs of type $Y(z)Y(w)$ are fixed by Jacobi identities $[X, Y, Y]$ and $[Y, Y, Y]$. The closed algebra, we get, is consistent with fusion

$$[\Psi] \times [\Psi] = [I]. \tag{10.7}$$

(No $[\Psi]$ term in the right hand side. I is the identity operator.)

The dimension $\frac{1}{2}$ and 1 fields Ψ and Φ so obtained are just free fermion and derivative of free boson respectively. Following the general procedure of [27] the free

28

fermion and the free boson can be decomposed from the algebra:

$$T = \widetilde{T} - \frac{1}{2}{:}\Psi\partial\Psi{:} + \frac{1}{2}{:}\Phi\Phi{:}\,,$$

$$G = \widetilde{G} + {:}\Psi\Phi{:}\,,$$

$$A = \frac{3+c}{12+c}\,\widetilde{A} - {:}\Psi\Omega{:} - \frac{15-c}{2\,(12+c)}{:}\Psi\partial\Psi{:}\,, \tag{10.8}$$

$$U = \frac{1}{\sqrt{15-c}\,\sqrt{21+4\,c}}\Big(2\,(15-c)\,\sqrt{21+4\,c}\,\sqrt{2\,c-3}\,\widetilde{U}$$

$$\qquad + 3\,(12+c)({:}\Psi\Sigma{:} - {:}\Phi\Omega{:}) + (15-c)({:}\Psi\partial\Phi{:} - 2{:}\Phi\partial\Psi{:})\Big)\,.$$

The operators $\widetilde{T}$, $\widetilde{G}$, $\widetilde{A}$, $\widetilde{U}$, Ω, Σ are commutative with Ψ and Φ, and form closed operator algebra. The new central charge is $\widetilde{c} = c - 3/2$. Conformal dimensions (with respect to $\widetilde{T}$) of generators of the new algebra are 2, $\frac{3}{2}$, 2, $\frac{5}{2}$, $\frac{3}{2}$, 2. After some redefinition of generators the algebra is found to coincide with $\mathcal{SW}(3/2, 3/2, 2)$ superconformal algebra of Ref. [10] (for the case of zero coupling $C^{\mathcal{M}}_{\mathcal{M}\mathcal{M}}$). At $\widetilde{c} = 10\frac{1}{2}$ the algebra becomes the G_2 algebra of Ref.[8], the symmetry algebra of manifolds of G_2 exceptional holonomy. The $\mathcal{SW}(3/2, 3/2, 2)$ algebra can be thought as a natural generalization of the G_2 exceptional holonomy algebra to any value of central charge.

We have shown that there is a connection between $\mathcal{SW}(3/2, 3/2, 2)$ and $\mathcal{SW}(3/2, 2)$ algebras:

$$\boxed{\mathcal{SW}(3/2, 3/2, 2)} \otimes \boxed{\substack{\text{free}\\\text{fermion}}} \otimes \boxed{\substack{\text{free}\\\text{boson}}} = \boxed{\substack{\mathcal{SW}(3/2, 2)\\\text{extended by}\\h = 1/2\ \text{multiplet}}} \tag{10.9}$$

In the case $\widetilde{c} = 10\frac{1}{2}$, $c = \widetilde{c} + 3/2 = 12$ the tensor product connects G_2 and $Spin(7)$ exceptional algebras discussed in [8].

Relation (10.9) restricts the unitarity of $\mathcal{SW}(3/2, 3/2, 2)$ algebra. Suppose there exists a unitary representation of $\mathcal{SW}(3/2, 3/2, 2)$ algebra at central charge $\widetilde{c}$. Multiplying it by some unitary representations of the free fermion and the free boson theories we get a unitary representation of extended $\mathcal{SW}(3/2, 2)$ algebra, which is necessarily unitary representation of $\mathcal{SW}(3/2, 2)$ algebra itself. This proves that unitary representations of $\mathcal{SW}(3/2, 3/2, 2)$ algebra can appear only at $\widetilde{c} = c_p^{(1)} - 3/2$ or $\widetilde{c} = c_p^{(2)} - 3/2$ or $\widetilde{c} = 4\frac{1}{2}$. The condition is necessary but not sufficient. Indeed the theory at $\widetilde{c} = 12/5 - 3/2 = 9/10$ is not unitary by the nonunitarity theorem [14, 15]. The extension of $\mathcal{SW}(3/2, 2)$ algebra, we got, corresponds to fusion (10.7), which may be inconsistent with unitarity.

We intend to study the representation theory of $\mathcal{SW}(3/2, 3/2, 2)$ superconformal algebra in another paper.

10.2 $\mathcal{SW}^2(1,2)$ algebra

The second extension we discuss is the extension of $\mathcal{SW}(3/2,2)$ by $\big|\,0\,,1\,\big\rangle$ representation, which belongs to continuous spectrum of all $\mathcal{SW}(3/2,2)$ unitary models. The extended algebra is $N=2$ super–W_3 algebra of Romans [11]. In the conventions of [2] the algebra is denoted by $\mathcal{SW}^2(1,2)$, meaning that the algebra consists of $N=2$ superconformal algebra and its dimension 2 multiplet (4 Virasoro primary fields of dimensions 2, $\frac{5}{2}$, $\frac{5}{2}$, 3). Romans has shown that $\mathcal{SW}^2(1,2)$ algebra contains $\mathcal{SW}(3/2,2)$ as a subalgebra. Here we go opposite way: we extend $\mathcal{SW}(3/2,2)$ to $\mathcal{SW}^2(1,2)$.

[J] is $\mathcal{SW}(3/2,2)$ conformal family of $\big|\,0\,,1\,\big\rangle$, and $J(w)$ is the field corresponding to its highest weight state. It obeys OPEs (5.2) with $h=1$ and $a=0$. The new fields, to be introduced, correspond to states $G_{-1/2}\big|\,0\,,1\,\big\rangle$, $U_{-3/2}\big|\,0\,,1\,\big\rangle$ and $U_{-3/2}\,G_{-1/2}\big|\,0\,,1\,\big\rangle$. $A_{-1}\big|\,0\,,1\,\big\rangle$ is null and $U_{-1/2}\big|\,0\,,1\,\big\rangle$ is proportional to $G_{-1/2}\big|\,0\,,1\,\big\rangle$. So the $[J]$ multiplet consists of 4 fields of dimensions 1, $\frac{3}{2}$, $\frac{5}{2}$, 3. Adding them to the list of dimensions of generators of $\mathcal{SW}(3/2,2)$ algebra we get the list of dimensions of generators of $\mathcal{SW}^2(1,2)$: 1, $\frac{3}{2}$, $\frac{3}{2}$, 2, 2, $\frac{5}{2}$, $\frac{5}{2}$, 3.

Unitary models of $\mathcal{SW}^2(1,2)$ fall to the same values of central charge $c=c_p^{(1)}$, $c=c_p^{(2)}$, $c=6$ as the unitary models of its $\mathcal{SW}(3/2,2)$ subalgebra. Romans [11] gives coset construction of $\mathcal{SW}^2(1,2)$ unitary minimal models by $\frac{SU(3)}{SU(2)\times U(1)}$ Kazama–Suzuki cosets [28, 29]. The ascending series models are represented by coset

$$\frac{SU(3)_k \times SO(4)_1}{SU(2)_{k+1} \times U(1)}, \tag{10.10}$$

and the descending series models are represented by

$$\frac{SU(2,1)_{-k-5} \times SO(4)_1}{SU(2)_{-k-4} \times U(1)}. \tag{10.11}$$

$SO(4)_1$ is decomposed to $SU(2)_1 \times SU(2)_1$. Then $SU(2)$ in the denominator is given by diagonal embedding to the direct sum of $SU(2)$ subalgebra of $SU(3)$, generated by $\{H^1, E^{\alpha_1+\alpha_2}, E^{-\alpha_1-\alpha_2}\}$, and the first $SU(2)$ in the $SO(4)$. (The embedding obviously gives the internal Virasoro algebra.) The generator of $U(1)$ in the denominator is $H^2 + \sqrt{3}\,J$, where H^2 is the second Cartan generator of $SU(3)$, commuting with $E^{\alpha_1+\alpha_2}$, and J is the Cartan generator of the second $SU(2)$ in the $SO(4)$.

It is noted in [11] that there exists a $\mathbb{Z}_2$ involution of $\mathcal{SW}^2(1,2)$ algebra (see (3.27) of [11]). The generators of $\mathcal{SW}^2(1,2)$, invariant under the involution, constitute the $\mathcal{SW}(3/2,2)$ subalgebra of $\mathcal{SW}^2(1,2)$. The field content of the $\mathcal{SW}^2(1,2)$ minimal models (10.10, 10.11) can be projected by the involution. One takes a $\mathbb{Z}_2$ invariant combination of $\mathcal{SW}^2(1,2)$ highest weight states and decomposes it to a sum

of the highest weight representations of $\mathcal{SW}(3/2,2)$ algebra. The projected models represent a nontrivial realization of $\mathcal{SW}(3/2,2)$ symmetry algebra. Every such a model contains the discrete spectrum of $\mathcal{SW}(3/2,2)$ algebra and infinite number of $\mathcal{SW}(3/2,2)$ representations from the continuous spectrum.

A Operator product expansions of $\mathcal{SW}(3/2,2)$ algebra

$$W = \frac{3\,(12+c)\,A - (15-c)\,T}{\sqrt{15-c}\,\sqrt{21+4\,c}}\,,$$

$$T(z)\,A(w) = A(z)\,A(w) = \frac{\frac{c\,(15-c)}{6\,(12+c)}}{(z-w)^4} + \frac{2\,A(w)}{(z-w)^2} + \frac{\partial A(w)}{z-w}\,, \tag{A.1}$$

$$A(z)\,G(w) = \frac{15-c}{3\,(12+c)}\left(\frac{\frac{3}{2}\,G(w)}{(z-w)^2} + \frac{\partial G(w) - \frac{\sqrt{21+4\,c}}{\sqrt{15-c}}\,U(w)}{z-w} \right), \tag{A.2}$$

$$A(z)\,U(w) = \frac{1}{(z-w)^3}\left(-\frac{\sqrt{15-c}\,\sqrt{21+4\,c}}{12+c}\,G(w) \right) +$$
$$\frac{1}{(z-w)^2}\left(\frac{87+5\,c}{6\,(12+c)}\,U(w) - \frac{\sqrt{15-c}\,\sqrt{21+4\,c}}{3\,(12+c)}\,\partial G(w) \right) +$$
$$\frac{1}{z-w}\left(\frac{(15-c)^{3/2}\,\partial^2 G(w)}{3\,(12+c)\,\sqrt{21+4\,c}} - \frac{(15-c)\,\partial U(w)}{3\,(12+c)} - \frac{54\,{:}AG{:}(w)}{\sqrt{15-c}\,\sqrt{21+4\,c}} \right), \tag{A.3}$$

$$G(z)\,G(w) = \frac{2\,c/3}{(z-w)^3} + \frac{2\,T(w)}{z-w}\,, \tag{A.4}$$

$$G(z)\,U(w) = \frac{\sqrt{15-c}}{\sqrt{21+4\,c}}\left(\frac{\frac{12\,(12+c)}{15-c}\,A(w) - 4\,T(w)}{(z-w)^2} + \frac{\frac{3\,(12+c)}{15-c}\,A(w) - T(w)}{z-w} \right), \tag{A.5}$$

$$U(z)\,U(w) = -\frac{2\,c}{(z-w)^5} +$$
$$\frac{1}{(z-w)^3}\left(-\frac{2\,(93+10\,c)}{21+4\,c}\,T(w) - \frac{12\,(12+c)\,(6+5\,c)}{(15-c)\,(21+4\,c)}\,A(w) \right) +$$
$$\frac{1}{(z-w)^2}\left(-\frac{93+10\,c}{21+4\,c}\,\partial T(w) - \frac{6\,(12+c)\,(6+5\,c)}{(15-c)\,(21+4\,c)}\,A(w) \right) +$$
$$\frac{1}{z-w}\left(-\frac{3\,(12+c)}{21+4\,c}\,\partial^2 T(w) + \frac{9\,(6-c)\,(12+c)}{(15-c)\,(21+4\,c)}\,\partial^2 A(w) - \right.$$
$$\left. \frac{324\,(12+c)\,{:}TA{:}(w)}{(15-c)\,(21+4\,c)} + \frac{27\,{:}G\partial G{:}(w)}{21+4\,c} + \frac{54\,{:}GU{:}(w)}{\sqrt{15-c}\,\sqrt{21+4\,c}} \right). \tag{A.6}$$

B List of unitary representations of $\mathcal{SW}(3/2,2)$ algebra

$r = \pm 1$ distinguish between $\big|\uparrow\big\rangle$ and $\big|\downarrow\big\rangle$ states of two dimensional Ramond representations. x stands for real positive number in the continuous spectrum expressions.

B.1 The unitary spectrum at $c = 6 - \dfrac{18}{p+1}$

NS sector

Discrete spectrum

a	h	$m,n \in \mathbb{Z}$
$\frac{(p\,m-(p+1)\,n)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(n-m)^2+(m-3\,n+1)(m-n-1)-m}{2\,(1+p)}$	$1 \le m \le p-1$ $m \le n \le \frac{p-1+m}{2}$
$\frac{(p\,m-(p+1)\,n)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(n-m)^2+(-m+3\,n+1)(-m+n-1)+m}{2\,(1+p)}$	$1 \le m \le p-2$ $m+1 \le n \le \frac{p+m}{2}$

Continuous spectrum

a	h	$m,n \in \mathbb{Z}$
$\frac{(p\,m-(p+1)\,n)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(m-n)^2+(m-3\,n+1)(m-n-1)-m}{2\,(1+p)} + x$	$1 \le m \le p-2$ $m \le n \le \frac{p-2+m}{2}$

Ramond sector

Discrete spectrum

a	h	$m,n \in \mathbb{Z}$
$\frac{m^2-1}{4\,p\,(p+1)}$	$\frac{p-2}{4\,(p+1)}$	$1 \le m \le p-1$
$\frac{\left(p\,m-(p+1)\left(n+\frac{1-r}{2}\right)\right)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(n-m)^2+(n-m)(1-m+3n)+p\,(n-m)+p/2-1}{2(1+p)}$	$1 \le m \le p-3$ $m+1 \le n \le \frac{p-1+m}{2}$
$\frac{\left(p\,m-(p+1)\left(n-\frac{1+r}{2}\right)\right)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(n-m)^2-(n-m)(1+m-3n)-p\,(n-m)+p/2-1}{2(1+p)}$	$1 \le m \le p-2$ $m+1 \le n \le \frac{p+m}{2}$

Continuous spectrum

a	h	$m,n \in \mathbb{Z}$
$\frac{\left(p\,m-(p+1)\left(n-\frac{1+r}{2}\right)\right)^2-1}{4\,p\,(p+1)}$	$\frac{p\,(n-m)^2-(n-m)(1+m-3n)-p\,(n-m)+p/2-1}{2(1+p)} + x$	$1 \le m \le p-3$ $m+1 \le n \le \frac{p-1+m}{2}$ and $2 \le m = n \le \frac{p+1}{2}$

B.2   The unitary spectrum at $c = 6 + \dfrac{18}{p}$

NS sector

Discrete spectrum

a	h	$m, n \in \mathbb{Z}$
0	0	
$\frac{(p(n+1)+2n+1)^2-1}{4p(p+1)}$	$\frac{p(n+1)^2+2n(n+1)-2p}{2p}$	$1 \le n \le \frac{p}{2}-1$
$\frac{(pm-(p+1)n)^2-1}{4p(p+1)}$	$\frac{p(n-m)^2+(n-m)(2m-1)+m+1}{2p}$	$1 \le n \le p-2$ $n+1 \le m \le \frac{p+n}{2}$
$\frac{(pm-(p+1)n)^2-1}{4p(p+1)}$	$\frac{p(n-m)^2+(-n+m)(-2m-1)-m+1}{2p}$	$1 \le n \le p-1$ $\frac{n}{2} \le m \le n-1$

Continuous spectrum

a	h	$m, n \in \mathbb{Z}$
0	x	
$\frac{(p(n-1)+2n-1)^2-1}{4p(p+1)}$	$\frac{(n-1)(p(n-1)+2n)}{2p} + x$	$2 \le n \le \frac{p}{2}$
$\frac{(pm-(p+1)n)^2-1}{4p(p+1)}$	$\frac{p(n-m)^2+(n-m)(2m-1)+m+1}{2p} + x$	$1 \le n \le p-2$ $n+1 \le m \le \frac{p+n}{2}$

Ramond sector

Discrete spectrum

a	h	$m, n \in \mathbb{Z}$
$\frac{p+3}{4p}$	$\frac{p+3}{4p}$	
$\frac{n^2-1}{4p(p+1)}$	$\frac{p+3}{4p}$	$1 \le n \le p-1$
$\frac{\left(p(m+\frac{1-r}{2})-(p+1)n\right)^2-1}{4p(p+1)}$	$\frac{p(m-n)^2-2m(m-n)+p(m-n)+p/2+3/2}{2p}$	$1 \le n \le p-3$ $n+1 \le m \le \frac{p-1+m}{2}$
$\frac{\left(p(m+\frac{1-r}{2})-(p+1)n\right)^2-1}{4p(p+1)}$	$\frac{p(m-n)^2+2(m-1)(1-m+n)+p(m-n)-3(p-1)/2}{2p}$	$1 \le n \le p-3$ $n+2 \le m \le \frac{p+1+m}{2}$

Continuous spectrum

a	h	$m, n \in \mathbb{Z}$
$\frac{\left(p(n+\frac{1-r}{2})+2n\right)^2-1}{4p(p+1)}$	$\frac{(2n-1)(2n+1-p)+2pn^2}{4p} + x$	$1 \le n \le \frac{p-1}{2}$
$\frac{\left(p(m+\frac{1-r}{2})-(p+1)n\right)^2-1}{4p(p+1)}$	$\frac{p(m-n)^2-2m(m-n)+p(m-n)+p/2+3/2}{2p} + x$	$1 \le n \le p-3$ $n+1 \le m \le \frac{p-1+m}{2}$ and $1 \le n = m \le \frac{p}{2}$

C Normal ordered product conventions

The normal ordered product is defined as the zero order term in OPE:

$$P(z)\, Q(w) = \text{singular terms} + :\!PQ\!:(w) + O(z-w)\,. \tag{C.1}$$

The mode expansion of $:\!PQ\!:$ can be easily calculated:

$$:\!PQ\!:_n = \sum_{m \le -\Delta_P} P_m Q_{n-m} + (-1)^{PQ} \sum_{m \ge -\Delta_P+1} Q_{n-m} P_m\,, \tag{C.2}$$

where Δ_P is the dimension of field $P(z)$ and $(-1)^{PQ}$ is equal to -1 only if both P and Q are fermionic operators. The expansion is valid for P bosonic ($m \in \mathbb{Z}$) or P fermionic of NS type ($m \in \mathbb{Z} + \frac{1}{2}$). If P is fermionic of Ramond type, then $m \in \mathbb{Z}$ but Δ_P is half–integer and uncertainty appears in the limits of the sums in (C.2).

If P is fermionic of Ramond type and Q is bosonic, one can express $:\!PQ\!:$ through $:\!QP\!:$ and then expand by use of (C.2). The problem appears then both P and Q are fermionic operators of Ramond type. The limits in the sums in the $:\!PQ\!:_n$ expansion should be chosen consistently. We rather fix the limits, but add some terms to the expansion:

$$:\!PQ\!:_n^{\mathrm{R}} = \sum_{m \le -\Delta_P+1/2} P_m Q_{n-m} - \sum_{m \ge -\Delta_P+3/2} Q_{n-m} P_m + \sum_i \gamma^{(i)}(c,n)\, S_n^{(i)}\,, \tag{C.3}$$

where $n, m \in \mathbb{Z}$, $S^{(i)}$ are operators entering to the singular part of OPE $P(z)\, Q(w)$, and $\gamma^{(i)}$ are some functions of the central charge and index n.

In the case of $\mathcal{SW}(3/2, 2)$ algebra we are interested in $:\!GU\!:$ and $:\!G\partial G\!:$ expansions:

$$:\!GU\!:_n^{\mathrm{R}} = \sum_{m \le -1} G_m U_{n-m} - \sum_{m \ge 0} U_{n-m} G_m + \gamma^{(1)} W_n\,, \tag{C.4}$$

$$:\!G\partial G\!:_n^{\mathrm{R}} = \sum_{m \le -1} G_m (\partial G)_{n-m} - \sum_{m \ge 0} (\partial G)_{n-m} G_m + \gamma^{(2)} \delta_{n,0} + \gamma^{(3)} L_n\,. \tag{C.5}$$

To fix the coefficients γ one can discuss commutator $[P_n, :\!TQ\!:_0]$. The commutator can be calculated in two ways. The first is to calculate the OPE $P(z)\,:\!TQ\!:(w)$, which will include $:\!PQ\!:(w)$. After that the commutator is obtained from the OPE by usual procedure. The second way is to expand first $:\!TQ\!:_0$ by use of (C.2), and then to calculate the commutator. By equating the results of two computations one fixes the expansion of $:\!PQ\!:_n^{\mathrm{R}}$. Sometimes it is easier to use instead of T another bosonic operator. For example, we get the $:\!GU\!:_n^{\mathrm{R}}$ expansion from $[G_n, :\!WG\!:_0]$ and the $:\!G\partial G\!:_n^{\mathrm{R}}$ expansion from $[G_n, :\!TG\!:_0]$. The relevant OPEs are:

$$G(z)\,:\!WG\!:(w) = \frac{\left(\frac{2}{3}c - 4\right) W(w)}{(z-w)^3} - \frac{3\,\partial W(w)}{(z-w)^2} +$$

$$+ \frac{2\,{:}TW{:}(w) - \partial^2 W(w) - {:}GU{:}(w)}{z - w}\,, \qquad \text{(C.6)}$$

$$G(z)\,{:}TG{:}(w) = \frac{3\,c}{(z-w)^5} + \frac{\left(\frac{2}{3}\,c + 5\right) T(w)}{(z-w)^3} + \frac{\frac{3}{2}\,\partial T(w)}{(z-w)^2} +$$

$$+ \frac{2\,{:}TT{:}(w) + \frac{1}{2}\,\partial^2 T(w) - \frac{1}{2}\,{:}G\partial G{:}(w)}{z - w}\,. \qquad \text{(C.7)}$$

And the γ coefficients in (C.4) and (C.5):

$$\gamma^{(1)} = \frac{3 + n}{2}\,, \qquad \gamma^{(2)} = \frac{5\,c}{64}\,, \qquad \gamma^{(3)} = \frac{9}{4} + n\,. \qquad \text{(C.8)}$$

References

[1] A. B. Zamolodchikov, "Infinite Additional Symmetries In Two-Dimensional Conformal Quantum Field Theory," Theor. Math. Phys. **65** (1985) 1205.

[2] P. Bouwknegt and K. Schoutens, "W symmetry in conformal field theory," Phys. Rept. **223** (1993) 183 [hep-th/9210010].

[3] V. A. Fateev and A. B. Zamolodchikov, "Conformal Quantum Field Theory Models In Two Dimensions Having Z(3) Symmetry," Nucl. Phys. **B280** (1987) 644.

[4] V. A. Fateev and S. L. Lukyakhov, "The Models Of Two-Dimensional Conformal Quantum Field Theory With Z(N) Symmetry," Int. J. Mod. Phys. **A3** (1988) 507.

[5] S. L. Lukyanov and V. A. Fateev, "Physics Reviews: Additional Symmetries And Exactly Soluble Models In Two-Dimensional Conformal Field Theory," *Chur, Switzerland: Harwood (1990) 117 p. (Soviet Scientific Reviews A, Physics: 15.2).*

[6] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, "Infinite conformal symmetry in two-dimensional quantum field theory," Nucl. Phys. **B241** (1984) 333.

[7] J. M. Figueroa-O'Farrill and S. Schrans, "The Conformal bootstrap and super W algebras," Int. J. Mod. Phys. A **7** (1992) 591.

J. M. Figueroa-O'Farrill and S. Schrans, "Extended superconformal algebras," Phys. Lett. **B257** (1991) 69.

[8] S. L. Shatashvili and C. Vafa, "Superstrings and manifold of exceptional holonomy," hep-th/9407025.

[9] S. L. Shatashvili and C. Vafa, "Exceptional magic," Nucl. Phys. Proc. Suppl. **41** (1995) 345.

[10] R. Blumenhagen, "Covariant construction of N=1 super W algebras," Nucl. Phys. **B381** (1992) 641.

[11] L. J. Romans, "The N=2 super W(3) algebra," Nucl. Phys. **B369** (1992) 403.

[12] K. Thielemans, "A Mathematica package for computing operator product expansions," Int. J. Mod. Phys. **C2** (1991) 787.

[13] J. M. Figueroa-O'Farrill, "A note on the extended superconformal algebras associated with manifolds of exceptional holonomy," Phys. Lett. **B392** (1997) 77 [hep-th/9609113].

[14] D. Friedan, Z. Qiu and S. Shenker, "Conformal Invariance, Unitarity And Two-Dimensional Critical Exponents," in *C83-11-10.1* Phys. Rev. Lett. **52** (1984) 1575.

[15] D. Friedan, S. Shenker and Z. Qiu, "Details Of The Nonunitarity Proof For Highest Weight Representations Of The Virasoro Algebra," Commun. Math. Phys. **107** (1986) 535.

[16] E. S. Fradkin and V. Y. Linetsky, "Results of the classification of superconformal algebras in two-dimensions," Phys. Lett. **B282** (1992) 352 [hep-th/9203045].

[17] E. S. Fradkin and V. Y. Linetsky, "Classification of superconformal and quasisuperconformal algebras in two-dimensions," Phys. Lett. **B291** (1992) 71 [hep-th/9207035].

[18] P. Ginsparg, "Applied Conformal Field Theory," HUTP-88-A054 *Lectures given at Les Houches Summer School in Theoretical Physics, Les Houches, France, Jun 28 - Aug 5, 1988.*

[19] S. Komata, K. Mohri and H. Nohara, "Classical and quantum extended superconformal algebra," Nucl. Phys. **B359** (1991) 168.

[20] S. Mallwitz, "On SW minimal models and N=1 supersymmetric quantum Toda field theories," Int. J. Mod. Phys. **A10**, 977 (1995) [hep-th/9405025].

[21] B. L. Feigin and D. B. Fuks, "Invariant Skew Symmetric Differential Operators On The Line And Verma Modules Over The Virasoro Algebra," Funct. Anal. Appl. **16** (1982) 114.

[22] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, "Superconformal Symmetry In Two-Dimensions," Phys. Lett. **B151** (1985) 31.

[23] G. Mussardo, G. Sotkov and M. Stanishkov, "Ramond Sector Of The Supersymmetric Minimal Models," Phys. Lett. **B195** (1987) 397.

[24] G. Mussardo, G. Sotkov and M. Stanishkov, "N=2 Superconformal Minimal Models," Int. J. Mod. Phys. **A4** (1989) 1135.

[25] V. S. Dotsenko and V. A. Fateev, "Conformal algebra and multipoint correlation functions in 2D statistical models," Nucl. Phys. **B240** (1984) 312.

[26] A. Schwimmer and N. Seiberg, "Comments On The N=2, N=3, N=4 Superconformal Algebras In Two-Dimensions," Phys. Lett. **B184** (1987) 191.

[27] P. Goddard and A. Schwimmer, "Factoring Out Free Fermions And Superconformal Algebras," Phys. Lett. **B214** (1988) 209.

[28] Y. Kazama and H. Suzuki, "New N=2 Superconformal Field Theories And Superstring Compactification," Nucl. Phys. **B321** (1989) 232.

[29] Y. Kazama and H. Suzuki, "Characterization Of N=2 Superconformal Models Generated By Coset Space Method," Phys. Lett. **B216** (1989) 112.